Scale-Invariant Subspace Detectors Based on First- and Second-Order Statistical Models

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***Abstract*—The problem is to detect a multi-dimensional source transmitting an unknown sequence of complex-valued symbols to a multi-sensor array. In some cases the channel subspace is known, and in others only its dimension is known. Should the unknown transmissions be treated as unknowns in a first-order statistical model, or should they be assigned a prior distribution that is then used to marginalize a first-order model for a second-order statistical model? This question motivates the derivation of subspace detectors for cases where the subspace is known, and for cases where only the dimension of the subspace is known. For three of these four models the GLR detectors are known, and they have been reported in the literature. But the GLR detector for the case of a known subspace and a second-order model for the measurements is derived for the first time in this paper. When the subspace is known, second-order generalized likelihood ratio (GLR) tests outperform first-order GLR tests when the spread of subspace eigenvalues is large, while first-order GLR tests outperform second-order GLR tests when the spread is small. When only the dimension of the subspace is known, second-order GLR tests outperform first-order GLR tests, regardless of the spread of signal subspace eigenvalues. For a dimension-1 source, first-order and second-order statistical models lead to equivalent GLR tests. This is a new finding.**

***Index Terms*—Detection, generalized likelihood ratio (GLR), likelihood, multi-sensor array, multivariate normal model (MVN), scale-invariant detector, subspace signals.**

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I. Introduction

BEGIN with a multisensor array consisting of L elements, and a measurement plan that records N independent measurements in time. These measurements may carry transmissions from a dimension-p source, and the detection problem is to determine whether or not they do. The basic question is this: should the unknown transmissions be treated as unknowns in a first-order statistical model, or should they be assigned a prior distribution that is then used to marginalize a first-order model for a second-order statistical model? This question motivates the derivation of subspace detectors for cases where the subspace is known, and for cases where only the dimension of the subspace is known.

There are many motivations for detecting dimension-p sources. In cognitive radio the problem is to determine whether a multi-user channel is vacant, so that an opportunistic user may use the channel without interferring with licensed users [1]. If the cognitive radio channel supports p independent single-antenna primary users or a single user transmitting p independent data streams, then the source transmission may be modeled as a dimension-p source [2]. In some cases the subspace is known to be spanned by time-, frequency-, or code-division multiplexes, and in other cases only the number of users is known. In modern radars and communication sets, transmissions may be drawn from a linear combination of known or unknown modes [3], [4]. In more established technologies like array processing for radar, sonar, and vibration analysis, transmissions from a single source are transmitted over a multipath or dispersive channel. At a receiving array the propagating wavefront may then be wrinkled, and modeled as a linear combination of wavefronts drawn from a p-dimensional basis, which may be known or unknown [5].

Generalized likelihood ratio (GLR) tests based on first-order statistical models, where an unknown signal is received through a basis for a known subspace have been derived for the past two decades for various signal and noise model assumptions [6]–[10]. These GLR tests have found numerous applications, such as radar and sonar [11], passive radar and source localization [12], and eavesdropper detection [13]. Second-order statistical models, where the unknown transmitted signal is assigned a Gaussian prior and then marginalized to yield binary detection problems on the structure of the covariance matrix, have been studied in the multivariate statistics literature [14]–[16]. More recently, GLR tests based on

second-order statistical models have been derived for multi-antenna sensing problems under various signal and noise models [2], [17]–[21]. These tests can be generalized to spacetime detection problems [22], [23], improper signals [24], or cyclostationary signals [25], [26]. The Gaussian prior is not the only alternative, yet a convenient one, and, for instance, [9] marginalizes over Haar measure on the Grassmann manifold of p-dimensional subspaces. In addition to the Gaussian prior on the source signal, a complex inverse Wishart prior can also be defined on the covariance matrix to derive a maximum a posteriori (MAP) estimator [27].

Despite the numerous GLR tests based on first-order and second-order statistical models, several questions remain unanswered: what is the second-order variation on the classical result in [28] by Anderson when the basis for a dimension-p subspace is known, and not just its dimension? Are there cases when a first-order model produces a GLR detector that is equivalent to the GLR detector that is produced by a second-order model? Are there problems for which one model is preferred over another, or is the answer to this question dependent on the number of array elements L, the source rank p, the sample support N, the signal-to-noise ratio (SNR), and the spread of source eigenvalues? These are the questions that motivate this work.

When only the dimension of the channel subspace is known, then numerical experiments indicate that the GLR derived from a second-order model outperforms the GLR derived from a first-order model. This is consistent with findings reported in [9] for a detector derived from a non-Gaussian prior distribution on the source signals. When the subspace is known, then the relative performance depends on the spread of the eigenvalues of the signal subspace. GLRs based on second-order models outperform GLRs based on first-order models when the eigenvalue spread of the source is large, while the GLR derived from a first-order model outperforms the GLR derived from a second-order model when the eigenvalue spread is small. For a dimension-1 source, we prove that first-order and second-order statistical models lead to equivalent generalized likelihood ratio (GLR) tests under both cases, known subspace and unknown subspace of known dimension.

Notation: In this paper, matrices are denoted by bold-faced upper case letters, column vectors are denoted by bold-faced lower case letters, and scalars are denoted by light-face lower case letters. The superscripts $(\cdot)^T$ and $(\cdot)^H$ denote transpose and Hermitian, respectively. The trace and determinant of a matrix $\mathbf{A}$ will be denoted, respectively, as $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$. The set $\mathbb{C}^L$ is the complex Euclidean space with the standard inner product and $GL(\mathbb{C}^p)$ denotes the general linear group of non-singular $p \times p$ matrices. An $n \times n$ identity matrix is denoted by $\mathbf{I}_n$ and $\mathbf{I}_n \otimes \mathbf{I}_L$ is a Kronecker product of identities. The notation $\langle \mathbf{H} \rangle$ denotes a subspace of the complex vector space $\mathbb{C}^L$ spanned by the columns of $\mathbf{H}$ and $\mathbf{P}_{\mathbf{H}}$ denotes the orthogonal projector onto $\langle \mathbf{H} \rangle$. The matrix $\mathrm{diag}(\mathbf{a})$ denotes a diagonal matrix whose diagonal is $\mathbf{a}$. The notation $\mathcal{CN}(0,1)$ denotes a complex Gaussian distribution with zero mean and unit variance and $\mathbf{x} \sim \mathcal{CN}_L(\mathbf{0}, \mathbf{R})$ denotes a complex Gaussian vector in $\mathbb{C}^L$ with zero mean and covariance $\mathbf{R}$. To say that a random matrix $\mathbf{X} \in \mathbb{C}^{L \times N}$ is normally distributed as $\mathbf{X} \sim \mathcal{CN}_{LN}(\mathbf{0}, \mathbf{\Sigma}_r \otimes \mathbf{\Sigma}_c)$, where $\mathbf{\Sigma}_r$ and $\mathbf{\Sigma}_c$ are respectively $N \times N$ and $L \times L$ positive definite matrices, is to say that $\mathbf{\Sigma}_r \otimes \mathbf{\Sigma}_c$ is the covariance of the column vector $\mathbf{x} = \mathrm{vec}(\mathbf{X})$. This column vector $\mathbf{x}$ is a stack of columns of the matrix $\mathbf{X}$.

II. Linear Measurement Model

The linear measurement model is

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}, \tag{1}$$

where $\mathbf{z} \in \mathbb{C}^L$ is the measurement at an L-element array, $\mathbf{H} \in \mathbb{C}^{L \times p}$ is a channel map from p sources to L array elements, and $\mathbf{x} \in \mathbb{C}^p$ are source transmissions. We assume in this work that $L > 1$. The channel matrix $\mathbf{H}$ may be expressed as $\mathbf{H} = \mathbf{U}\mathbf{A}$, where $\mathbf{U}$ is an arbitrarily-chosen unitary basis for the channel subspace $\langle \mathbf{H} \rangle$, and $\mathbf{A} \in GL(\mathbb{C}^p)$ is an unknown non-singular $p \times p$ map from the sources into the subspace $\langle \mathbf{H} \rangle$. The noise $\mathbf{v}$ is modeled as a complex Gaussian vector with zero mean and covariance $\sigma^2 \mathbf{I}_L$: $\mathbf{v} \sim \mathcal{CN}_L(\mathbf{0}, \sigma^2 \mathbf{I}_L)$, with σ^2 unknown. The signal $\mathbf{x}$ may be treated as an unknown parameter, or as a random parameter, distributed as $\mathbf{x} \sim \mathcal{CN}_p(\mathbf{0}, \mathbf{C}_{xx})$, where $\mathbf{C}_{xx}$ is an unknown $p \times p$ covariance matrix. Given N independent and identically distributed (i.i.d.) realizations, the measurement equation can be written as

$$\mathbf{Z} = \mathbf{H}\mathbf{X} + \mathbf{V} = \mathbf{U}\mathbf{A}\mathbf{X} + \mathbf{V}, \tag{2}$$

where $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_N] \in \mathbb{C}^{p \times N}$, and $\mathbf{Z}$ and $\mathbf{V}$ are, respectively, the measurement and noise matrices. They are structured as $\mathbf{X}$ is structured. This model applies to multivariate normal noise with covariance $\sigma^2 \mathbf{\Sigma} > \mathbf{0}$, with $\mathbf{\Sigma}$ known, in which case the basis $\mathbf{U}$ and measurement $\mathbf{Z}$ are pre-whitened by $\mathbf{\Sigma}^{-1/2}$.

When $\mathbf{A}$ is given the SVD $\mathbf{A} = \mathbf{F}\mathbf{K}\mathbf{G}^H$, then the signal component of the measurement matrix may be written $(\mathbf{U}\mathbf{F})\mathbf{K}(\mathbf{G}^H\mathbf{X})$. So it is as if the symbols $\mathbf{G}^H\mathbf{X}$ are scaled by the diagonal matrix $\mathbf{K}$ and received on the frame $\mathbf{U}\mathbf{F}$. Even if the receiver knows a frame $\mathbf{U}$ for the subspace $\langle \mathbf{H} \rangle$, it does not know the orthogonal matrix $\mathbf{F}$, so there is no matching to $\mathbf{U}\mathbf{F}$. Only matching to $\mathbf{U}$.

A. First-Order Measurement Model

In a first-order model for the measurements, the signal matrix $\mathbf{X}$ in (2) is assumed to be unknown, with no prior distribution assigned. Since $\mathbf{A}$ and $\mathbf{X}$ are both unknown and unconstrained, $\mathbf{A}\mathbf{X}$ may be treated as the unknown $\mathbf{X}$. Then, the binary detection problem in the first-order model is

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{Z} \sim \mathcal{CN}_{LN}\left(\mathbf{0}, \sigma^2 \mathbf{I}_N \otimes \mathbf{I}_L\right), \\ \mathcal{H}_1 &: \mathbf{Z} \sim \mathcal{CN}_{LN}\left(\mathbf{U}\mathbf{X}, \sigma^2 \mathbf{I}_N \otimes \mathbf{I}_L\right), \end{aligned} \tag{3}$$

where the unknown signal $\mathbf{X}$ appears in the mean value of the measurements under the alternative hypothesis $\mathcal{H}_1$. The likelihood function for $\mathbf{X}$ and σ^2, given $\mathbf{Z}$, is

$$\ell_1 = \frac{1}{(\pi^L \sigma^2)^N} \exp\left\{ -\frac{\mathrm{tr}\left[(\mathbf{Z} - \mathbf{U}\mathbf{X})(\mathbf{Z} - \mathbf{U}\mathbf{X})^H\right]}{\sigma^2} \right\}. \tag{4}$$

B. Second-Order Measurement Model

When the linear measurement model $\mathbf{z} = \mathbf{U}\mathbf{A}\mathbf{x} + \mathbf{v}$ is marginalized with respect to normally distributed source symbols, $\mathbf{x} \sim \mathcal{CN}_p(\mathbf{0}, \mathbf{C}_{xx})$, the result is the second-order model

$$\mathbf{z} \sim \mathcal{CN}_L(\mathbf{0}, \mathbf{U}\mathbf{A}\mathbf{C}_{xx}\mathbf{A}^H\mathbf{U}^H + \sigma^2\mathbf{I}_L), \tag{5}$$

where the covariance matrix of the random signal $\mathbf{C}_{xx}$ appears in the covariance matrix of the measurement under $\mathcal{H}_1$. The measurements under $\mathcal{H}_0$ and $\mathcal{H}_1$ are i.i.d. realizations of zero-mean multivariate normal (MVN) random vectors with covariance matrices $\mathbf{R}_0 = \sigma^2\mathbf{I}_L$ and $\mathbf{R}_1 = \mathbf{U}\mathbf{A}\mathbf{C}_{xx}\mathbf{A}^H\mathbf{U}^H + \sigma^2\mathbf{I}_L$, respectively. Since the $p \times p$ covariance matrix $\mathbf{C}_{xx}$ is unknown, the matrix $\mathbf{A}\mathbf{C}_{xx}\mathbf{A}^H$ is also unknown. Therefore we may write, without loss of generality, $\mathbf{R}_1 = \sigma^2(\mathbf{U}\mathbf{Q}\mathbf{U}^H + \mathbf{I}_L)$, where $\mathbf{Q}$ is an unknown positive semidefinite matrix. The detection problem under a second-order measurement model amounts to testing the structure of the covariance matrix of the observations:

$$\begin{aligned}\mathcal{H}_0 &: \mathbf{Z} \sim \mathcal{CN}_{LN}\left(\mathbf{0}, \sigma^2\mathbf{I}_N \otimes \mathbf{I}_L\right),\\ \mathcal{H}_1 &: \mathbf{Z} \sim \mathcal{CN}_{LN}\left(\mathbf{0}, \sigma^2\mathbf{I}_N \otimes \left(\mathbf{U}\mathbf{Q}\mathbf{U}^H + \mathbf{I}_L\right)\right).\end{aligned} \tag{6}$$

The likelihood function for $\mathbf{R} = \mathbf{R}_i, (i = 0, 1)$, given $\mathbf{Z}$ is

$$\ell_2 = \frac{1}{(\pi^L \det(\mathbf{R}))^N} \exp\{-N\mathrm{tr}(\mathbf{R}^{-1}\mathbf{S})\} \tag{7}$$

where $\mathbf{S} = N^{-1}\mathbf{Z}\mathbf{Z}^H$ is the sample covariance matrix.

C. GLR Detectors

In a likelihood theory [29], [30], the unknown parameters under each hypothesis are replaced by their maximum likelihood (ML) estimates. In this work, we do not consider the case that both $\mathbf{U}$ and $\mathbf{A}$ are known, i.e., $\mathbf{H}$ is known, as this case applies only to coherent communication. It almost never applies to radar or sonar or any other imaging system where there is no phase lock. When only the basis $\mathbf{U}$ is known, we say the problem is to detect signal transmissions in a *known subspace* $\langle\mathbf{H}\rangle$. When $\mathbf{U}$ is unknown, but the dimension of $\mathbf{H} = \mathbf{U}\mathbf{A}$ is known, we say the problem is to detect signal transmissions in an *unknown subspace of known dimension*. There are several cases to be considered in this paper.

1) The basis $\mathbf{U}$ is known, which is equivalent to saying only the subspace $\langle\mathbf{H}\rangle$ is known. If the source transmission is treated as a sequence of i.i.d. random vectors, then $\mathbf{A}\mathbf{C}_{xx}\mathbf{A}^H$, the covariance matrix of $\mathbf{A}\mathbf{x}$, may be treated only as an unknown and positive semidefinite matrix, as any special structure imposed on $\mathbf{C}_{xx}$ will be washed out by $\mathbf{A}$. This case leads to a new detector not previously reported in the literature. If $\mathbf{A}\mathbf{x}$ is treated as an unknown parameter in the mean of the measurements, the resulting detector is the multipulse or multi-snapshot generalization of the scale-invariant matched subspace detector [6], which has been reported in [31].
2) Neither $\mathbf{U}$ nor $\mathbf{A}$ is known, but it is known that the unknown channel matrix $\mathbf{H} = \mathbf{U}\mathbf{A}$ has full rank p, which is equivalent to saying only the dimension of the subspace $\langle\mathbf{H}\rangle$ is known. In this case the received transmissions $\mathbf{U}\mathbf{A}\mathbf{x}$ are known only to consist of visits to an unknown subspace of dimension p. These visits may be treated as unknown, or they may be modeled as i.i.d. visits with rank-p unknown covariance $\mathbf{A}\mathbf{C}_{xx}\mathbf{A}^H$. If treated as unknown, the resulting detector is the generalization of the matched direction detector [8] reported in [9]. If treated as a sequence of random vectors, then the resulting detector is the detector reported by [2], based on Anderson's fundamental result [28].

III. Detectors for Signals in a Known Subspace

In this case, a basis $\mathbf{U}$ for the subspace $\langle\mathbf{H}\rangle$ is known, which is to say the unitary basis $\mathbf{U}$ is known, but the map $\mathbf{A}$ is unknown. In Section III-A we derive the GLR under a second-order model for the measurements, which has not been reported in the literature. In Section III-B we review the GLR under a first-order model.

A. GLR for a Second-Order Measurement Model

The detection problem amounts to testing the structure of the covariance matrix of the observations as stated in (6), and repeated here for convenience:

$$\begin{aligned}\mathcal{H}_0 &: \mathbf{Z} \sim \mathcal{CN}_{LN}\left(\mathbf{0}, \sigma^2\mathbf{I}_N \otimes \mathbf{I}_L\right),\\ \mathcal{H}_1 &: \mathbf{Z} \sim \mathcal{CN}_{LN}\left(\mathbf{0}, \sigma^2\mathbf{I}_N \otimes \left(\mathbf{U}\mathbf{Q}\mathbf{U}^H + \mathbf{I}_L\right)\right).\end{aligned}$$

We adopt a GLR approach, so the problem is to maximize (7) with respect to the unknown parameters under each hypothesis. To this end, we argue that to maximize likelihood is to maximize the monotone function of the likelihood

$$\mathcal{L} = \log\det(\mathbf{R}^{-1}\mathbf{S}) - \mathrm{tr}(\mathbf{R}^{-1}\mathbf{S})$$

under the constraint that $\mathbf{R} \in \mathcal{R}_i$, where $\mathcal{R}_i$ is a set determining the structure of the covariance matrix. More precisely, under $\mathcal{H}_0$ and $\mathcal{H}_1$ the covariance matrix belongs to the following sets

$$\begin{aligned}\mathcal{R}_0 &= \{\mathbf{R} = \sigma^2\mathbf{I}_L \mid \sigma^2 > 0\}\\ \mathcal{R}_1 &= \{\mathbf{R} = \sigma^2\left(\mathbf{U}\mathbf{Q}\mathbf{U}^H + \mathbf{I}_L\right) \mid \mathbf{Q} \succ \mathbf{0},\ \sigma^2 > 0\}\end{aligned}$$

which are both cones, since for any $\mathbf{R} \in \mathcal{R}_i, i = 0, 1$, and for any $a > 0$, we have $a\mathbf{R} \in \mathcal{R}_i, i = 0, 1$.

A lemma proved in [32] shows that in this case the covariance that maximizes the likelihood under each hypothesis satisfies the constraint $\mathrm{tr}(\hat{\mathbf{R}}^{-1}\mathbf{S}) = L$. Thus, the GLR is a ratio of determinants,

$$\Lambda_2 = \frac{\det(\hat{\mathbf{R}}_0)}{\det(\hat{\mathbf{R}}_1)}, \tag{8}$$

where $\hat{\mathbf{R}}_0$ and $\hat{\mathbf{R}}_1$ are the maximum likelihood (ML) estimates under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. Under $\mathcal{H}_0$, the ML estimate is

$$\hat{\mathbf{R}}_0 = \frac{\mathrm{tr}(\mathbf{S})}{L}\mathbf{I}_L. \tag{9}$$

Under $\mathcal{H}_1$, the ML estimates $\hat{\mathbf{Q}}$ and $\hat{\sigma}^2$ do not generally admit closed-form solutions. In the following paragraphs, we address this problem by deriving a solution for σ^2 that depends on the unknown covariance $\mathbf{Q}$, and then proposing an iterative algorithm for finding a maximizing solution for $\mathbf{Q}$.

The covariance under $\mathcal{H}_1$ is

$$\mathbf{R}_1 = \sigma^2 \left(\mathbf{U}\mathbf{Q}\mathbf{U}^H + \mathbf{I}_L\right), \tag{10}$$

where recall that $\mathbf{U}$ is a unitary basis for the known subspace $\langle \mathbf{H} \rangle$, and $\mathbf{Q}$ is an unknown non-negative definite matrix. Give $\mathbf{Q}$ the eigenvalue decomposition $\mathbf{Q} = \mathbf{F}\mathbf{\Lambda}\mathbf{F}^H$, where $\mathbf{\Lambda} = \text{diag}(\lambda_1, \ldots, \lambda_p)$, with $\lambda_1 \geq \cdots \geq \lambda_p$. Hence, the determinant and inverse of $\mathbf{R}_1$ may be written as

$$\begin{aligned} \det(\mathbf{R}_1) &= (\sigma^2)^L \det\left(\mathbf{U}\mathbf{Q}\mathbf{U}^H + \mathbf{I}_L\right) \\ &= (\sigma^2)^L \prod_{i=1}^{p} (\lambda_i + 1), \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathbf{R}_1^{-1} &= \frac{1}{\sigma^2}\left(\mathbf{I}_L - \mathbf{U}\left(\mathbf{Q}^{-1} + \mathbf{I}_p\right)^{-1}\mathbf{U}^H\right) \\ &= \frac{1}{\sigma^2}\left(\mathbf{I}_L - \mathbf{U}\mathbf{F}\mathbf{D}\mathbf{F}^H\mathbf{U}^H\right), \end{aligned} \tag{12}$$

where

$$\mathbf{D} = \text{diag}\left(\frac{\lambda_1}{\lambda_1 + 1}, \ldots, \frac{\lambda_p}{\lambda_p + 1}\right). \tag{13}$$

Discarding constant terms, the log-likelihood under $\mathcal{H}_1$ may be written as

$$\begin{aligned} \mathcal{L}(\mathbf{\Lambda}, \mathbf{F}, \sigma^2) = &-L\log(\sigma^2) - \sum_{i=1}^{p}\log(\lambda_i + 1) \\ &- \frac{1}{\sigma^2}\left[\text{tr}(\mathbf{S}) - \text{tr}\left(\mathbf{D}\mathbf{F}^H\mathbf{S}_\mathbf{H}\mathbf{F}\right)\right], \end{aligned} \tag{14}$$

where $\mathbf{S}_\mathbf{H} = \mathbf{U}^H\mathbf{S}\mathbf{U}$ is a quadratic form in an arbitrarily-chosen unitary basis $\mathbf{U}$ for the known subspace $\langle \mathbf{H} \rangle$, and the measured sample covariance $\mathbf{S}$. Taking the derivative of (14) with respect to σ^2 and equating it to zero, the ML estimate of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{L}\left[\text{tr}(\mathbf{S}) - \text{tr}\left(\mathbf{D}\mathbf{F}^H\mathbf{S}_\mathbf{H}\mathbf{F}\right)\right]. \tag{15}$$

Substituting this value into (14), the compressed log-likelihood becomes

$$\begin{aligned} \mathcal{L}(\mathbf{\Lambda}, \mathbf{F}, \hat{\sigma}^2) = &-L\log\left[\text{tr}(\mathbf{S}) - \text{tr}\left(\mathbf{D}\mathbf{F}^H\mathbf{S}_\mathbf{H}\mathbf{F}\right)\right] \\ &- \sum_{i=1}^{p}\log(\lambda_i + 1). \end{aligned} \tag{16}$$

The ML estimate of $\mathbf{F}$ is obtained by maximizing (16) or, equivalently, solving

$$\max_{\mathbf{F}^H\mathbf{F} = \mathbf{I}_p} \ \text{tr}\left(\mathbf{D}\mathbf{F}^H\mathbf{S}_\mathbf{H}\mathbf{F}\right). \tag{17}$$

The following Lemma from [33] gives us the maximum of (17) as a function of $\mathbf{D}$ and the eigenvalues of $\mathbf{S}_\mathbf{H}$.

Lemma 1: (Proposition A.2.a of [33], pp. 786): Let $\mathbf{A} = \mathbf{G}\,\text{diag}(\mu_1, \ldots, \mu_p)\,\mathbf{G}^H$ be a $p \times p$ Hermitian matrix with eigenvalues $\mu_1 \geq \cdots \geq \mu_p$, and let $\mathbf{B} = \mathbf{J}\,\text{diag}(\beta_1, \ldots, \beta_p)\,\mathbf{J}^H$ be a $p \times p$ Hermitian matrix with eigenvalues $\beta_1 \geq \cdots \geq \beta_p$, then

$$\max_{\mathbf{F}^H\mathbf{F} = \mathbf{I}_p} \ \text{tr}\left(\mathbf{F}^H\mathbf{A}\mathbf{F}\mathbf{B}\right) = \sum_{i=1}^{p}\mu_i\beta_i, \tag{18}$$

and the maximum is attained at $\hat{\mathbf{F}} = \mathbf{G}\mathbf{J}^H$.

Taking $\mathbf{A} = \mathbf{S}_\mathbf{H}$ and $\mathbf{B} = \mathbf{D}$ in the previous lemma, we obtain that the maximum value of $\text{tr}(\mathbf{D}\mathbf{F}^H\mathbf{S}_\mathbf{H}\mathbf{F})$ when evaluated at the ML estimate $\hat{\mathbf{F}}$ is

$$\text{tr}\left(\mathbf{D}\hat{\mathbf{F}}^H\mathbf{S}_\mathbf{H}\hat{\mathbf{F}}\right) = \sum_{i=1}^{p}\frac{\mu_i\lambda_i}{\lambda_i + 1}, \tag{19}$$

where $\mu_1 \geq \cdots \geq \mu_p$ are now the eigenvalues of $\mathbf{S}_\mathbf{H}$. Substituting (19) into (16), the compressed log-likelihood becomes a function solely of the eigenvalues of $\mathbf{Q}$:

$$\begin{aligned} \mathcal{L}(\mathbf{\Lambda}, \hat{\mathbf{F}}, \hat{\sigma}^2) = &-L\log\left(\text{tr}(\mathbf{S}) - \sum_{i=1}^{p}\frac{\mu_i\lambda_i}{\lambda_i + 1}\right) \\ &- \sum_{i=1}^{p}\log(\lambda_i + 1). \end{aligned} \tag{20}$$

This function is not concave. Therefore, an iterative algorithm is needed to maximize it with respect to the eigenvalues of $\mathbf{Q}$. A particularly convenient choice is to apply a cyclic optimization algorithm that, at each iteration, optimizes the value of a single eigenvalue assuming that the estimates of the other eigenvalues are fixed. Let us denote the estimate of λ_k at the lth iteration as $\hat{\lambda}_k^{(l)}$. At each iteration we sequentially update all p eigenvalues starting from $k = 1$ till $k = p$. Then, to update the k-th eigenvalue at iteration $l + 1$ the rest of eigenvalues take fixed values: $\hat{\lambda}_1^{(l+1)}, \ldots, \hat{\lambda}_{k-1}^{(l+1)}, \hat{\lambda}_{k+1}^{(l)}, \ldots \hat{\lambda}_p^{(l)}$. In this way, (20) becomes a univariate concave function whose maximum can be obtained in closed form as

$$\hat{\lambda}_k^{(l+1)} = \left(\frac{\gamma_k^{(l)} - L\mu_k}{\mu_k - \gamma_k^{(l)}}\right)^+, \tag{21}$$

where

$$\gamma_k^{(l)} = \text{tr}(\mathbf{S}) - \sum_{i=1}^{k-1}\frac{\mu_i\hat{\lambda}_i^{(l+1)}}{\hat{\lambda}_i^{(l+1)} + 1} - \sum_{i=k+1}^{p}\frac{\mu_i\hat{\lambda}_i^{(l)}}{\hat{\lambda}_i^{(l)} + 1}, \tag{22}$$

and $(x)^+ = \max(x, 0)$. A convenient initialization point is $\hat{\lambda}_k^{(0)} = 0$, $\forall k$. Iterations are stopped when the difference in the value of the compressed log-likelihood in two consecutive iterations is less than a prescribed threshold.

This iterative method, shown in Algorithm 1, can be framed as a block coordinate ascent method with cyclic update rule. At every step, the subproblem for each eigenvalue is solved for its unique global optimal solution and, therefore, according to [34], the proposed method converges to a stationary point of (20).

Once the ML estimates of the eigenvalues $\hat{\lambda}_i$ of $\mathbf{Q}$ have been obtained, the GLR is

$$\Lambda_2 = \frac{\text{tr}(\mathbf{S})^L}{\left(\text{tr}(\mathbf{S}) - \sum_{i=1}^{p}\frac{\hat{\lambda}_i\mu_i}{\hat{\lambda}_i + 1}\right)^L \prod_{i=1}^{p}(\hat{\lambda}_i + 1)}. \tag{23}$$

Algorithm 1: ML Estimates of the Eigenvalues of $\mathbf{Q}$.

Initialize: $l = 0, \hat{\lambda}_k^{(0)} = 0$
repeat
  for $k = 1$ to p **do**
    Compute $\gamma_k^{(l)}$ as in (22)
    Estimate $\hat{\lambda}_k^{(l+1)}$ using (21)
  end for
  $l = l + 1$
until convergence

There can be no closed-form expression for the distribution of Λ_2 under $\mathcal{H}_0$, since it is a function of an iteration that stops at some value of K. So certainly one cannot say that a threshold may be set to achieve a specified probability of false alarm using a known distribution. But suppose Λ_2 is computed using measurement $\mathbf{Z}$ and then replace this measurement by $\beta\mathbf{Z}$. It is as if the additive noise has unknown variance $\beta^2\sigma^2$, rather than σ^2. However, the sequence of iterates for $\hat{\lambda}_k^{(l)}$ in (21) is invariant to scale and therefore so also is the statistic Λ_2 under $\mathcal{H}_0$. In summary, since Λ_2 is scale-invariant, it is a constant false alarm rate (CFAR) detector with respect to measurement scaling.

B. GLR for a First-Order Measurement Model

In a first-order model for the measurements, the GLR may be considered as a multipulse generalization of the matched subspace detector [6], which has been reported in [31]:

$$\Lambda_1 = \frac{\text{tr}(\mathbf{Z}^H\mathbf{P}_\mathbf{H}\mathbf{Z})}{\text{tr}(\mathbf{Z}^H\mathbf{Z})}, \qquad 0 \le \Lambda_1 \le 1, \tag{24}$$

where $\mathbf{P}_\mathbf{H} = \mathbf{U}\mathbf{U}^H$ is the orthogonal projector onto $\langle\mathbf{H}\rangle$. As this equation shows, the GLR measures the fraction of total energy that lies in the subspace $\langle\mathbf{H}\rangle$. This is a coherence detector. The numerator matches the sample covariance matrix $\mathbf{Z}\mathbf{Z}^H$ to $\mathbf{P}_\mathbf{H}$, which would be the covariance matrix of a rank-p signal source of uncorrelated components, each of equal variance. This is only an interpretation, as no such assumption has been made in the derivation of the detector.

The detector (24) can be expressed as

$$\Lambda_1 = \frac{\text{tr}(\mathbf{Z}^H\mathbf{P}_\mathbf{H}\mathbf{Z})}{\text{tr}(\mathbf{Z}^H\mathbf{P}_\mathbf{H}\mathbf{Z}) + \text{tr}(\mathbf{Z}^H\mathbf{P}_\mathbf{H}^\perp\mathbf{Z})} = \frac{X}{X+Y}, \tag{25}$$

where $\mathbf{P}_\mathbf{H}^\perp$ is the projection onto the orthogonal complement of $\langle\mathbf{H}\rangle$. Under $\mathcal{H}_0$, X and Y are independent chi-squared random variables with $2Np$ and $2N(L-p)$ degrees of freedom, respectively [35]. Therefore, (24) follows under the null a beta distribution[1] with parameters Np and $N(L-p)$:

$$\Lambda_1 \sim \text{Beta}(Np, N(L-p)).$$

[1]A random variable, r, is said to be beta-distributed with parameters α and β, $r \sim \text{Beta}(\alpha, \beta)$, if its probability density function is

$$p(r) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}, \qquad 0 \le r \le 1;$$

where $\Gamma(x)$ is the Gamma function.

Remark 1 (The case $p = 1$): Specializing the GLR Λ_1 in (24) to $p = 1$, we have

$$\Lambda_1 = \frac{\mathbf{u}^H\mathbf{S}\mathbf{u}}{\text{tr}(\mathbf{S})} = \frac{\mu_1}{\text{tr}(\mathbf{S})}, \qquad 0 \le \Lambda_1 \le 1, \tag{26}$$

which is invariant to right unitary transformation of the basis vector $\mathbf{u}$ to the basis vector $\mathbf{u}q$, with q unimodular. For the GLR Λ_2 the ML estimate of $\hat{\lambda}$ in (21) when $p = 1$ can be obtained in closed-form as

$$\hat{\lambda} = \begin{cases} \frac{L\Lambda_1 - 1}{1-\Lambda_1}, & 1/L < \Lambda_1 \le 1, \\ 0, & 0 \le \Lambda_1 \le 1/L. \end{cases} \tag{27}$$

Plugging this estimate into (23), Λ_2 can be expressed in terms of Λ_1 as

$$\Lambda_2 = \begin{cases} \frac{1}{L}\left(1 - \frac{1}{L}\right)^{L-1} \frac{1}{\Lambda_1(1-\Lambda_1)^{L-1}}, & \Lambda_1 > \frac{1}{L}, \\ 1, & \Lambda_1 \le \frac{1}{L}. \end{cases} \tag{28}$$

This shows that Λ_2 is constant at 1 for $0 \le \Lambda_1 \le 1/L$, and monotonically increasing from there to infinity for $1/L < \Lambda_1 \le 1$. To compare Λ_2 to a threshold of 1 would be to always detect, and therefore to operate in the Northeast corner of its ROC curve, where $P_d = 1 = P_{fa}$. Therefore every decision threshold for Λ_2 will be greater than 1 and for any such threshold there will be a corresponding threshold for Λ_1 that may be read off the monotone curve relating Λ_2 to Λ_1. This makes the detectors Λ_2 and Λ_1 equivalent in the case of a dimension-one subspace ($p = 1$).

Certainly, the equivalence between dimension-one GLR detectors derived from first-order and second-order MVN models seems to happen only for the case of spatially white noise. A similar equivalence will not occur under other noise models or in other scenarios. For example, it is shown in [36] that when the noise has an arbitrary covariance matrix and there is a set of noise-only training data available, the GLR detectors for first-order and second-order models (or conditional and unconditional models, as referred to in [36]) are different.

IV. Detectors for Signals in an Unknown Subspace of Known Dimension

A. GLR for a Second-Order Measurement Model

In this case, the covariance matrix $\mathbf{U}\mathbf{A}\mathbf{C}_{xx}\mathbf{A}^H\mathbf{U}^H$ is an unknown rank-p covariance matrix. The GLR for this model when the noise variance is unknown is derived from Anderson's fundamental result [28] as reported in [2]

$$\Omega_2 = \frac{\left(\frac{\text{tr}(\mathbf{S})}{L}\right)^L}{\left(\prod_{i=1}^p \lambda_i(\mathbf{S})\right)\left(\frac{\sum_{i=p+1}^L \lambda_i(\mathbf{S})}{L-p}\right)^{L-p}}, \tag{29}$$

which is a ratio of determinants $\Omega_2 = \frac{\det(\hat{\mathbf{R}}_0)}{\det(\hat{\mathbf{R}}_1)}$. The ML estimate under the alternative hypothesis, $\hat{\mathbf{R}}_1$, was derived by Anderson in [28], and $\hat{\mathbf{R}}_0 = \frac{\text{tr}(\mathbf{S})}{L}\mathbf{I}_L$ is a standard result. The test for dimension-one signals under a second-order MVN has also been

studied extensively in the cognitive radio literature [17]–[19]. When $p \geq L-1$, the GLR in (29) reduces to the well-known sphericity test [37]. The exact null distribution for the sphericity test for the case of two sensors, $L=2$, and three sensors, $L=3$, with an arbitrary number of samples N can be found in [38] and [39], respectively. For $L>3$, exact null distributions are quite complicated expressions involving the Meijer G-function and hence appear to be of limited usefulness. Approximate expressions for the null distribution of the sphericity test based on fitting some known moments can be found in [40]. When $p<L-1$ closed-form expressions for the null distribution are not known.

Other variants and generalizations of this result have appeared in the literature. For example, the rank contrained ML estimate, $\hat{\mathbf{R}}_1$, when the noise variance is known or when only a lower bound on the noise power is available is obtained by thresholding the eigenvalues of the sample covariance matrix (29), as shown in [21]. When lower and upper bounds on the noise level are available and additional persymmetric constraints are enforced on the solution, the ML estimate of the covariance matrix (or MAP estimate if a prior is also given to $\mathbf{R}_1$) is derived in [27]. The corresponding GLRs can readily be obtained from these ML estimates.

B. GLR for a First-Order Measurement Model

In this case, only the dimension of the signal subspace is known. The GLR is the extension of [8] reported in [9]:

$$\Omega_1 = \frac{\sum_{i=1}^{p}\lambda_i(\mathbf{Z}\mathbf{Z}^H)}{\mathrm{tr}(\mathbf{Z}^H\mathbf{Z})}, \qquad \frac{1}{L} \leq \Omega_1 \leq 1, \tag{30}$$

where $\lambda_i(\mathbf{Z}\mathbf{Z}^H)$ denotes the ith eigenvalue of $\mathbf{Z}\mathbf{Z}^H$. This, too, is a coherence detector, where a known subspace is replaced by the span of the p dominant eigenvectors $\mathbf{F}_p$ of the sample covariance matrix. That is, with $\mathbf{Z}\mathbf{Z}^H = \mathbf{F}\mathbf{\Lambda}\mathbf{F}^H$, the detector may be written

$$\Omega_1 = \frac{\mathrm{tr}(\mathbf{Z}\mathbf{F}_p\mathbf{F}_p^H\mathbf{Z}^H)}{\mathrm{tr}(\mathbf{Z}^H\mathbf{Z})} = \frac{\mathrm{tr}(\mathbf{Z}\mathbf{P}_p\mathbf{Z}^H)}{\mathrm{tr}(\mathbf{Z}^H\mathbf{Z})}, \tag{31}$$

where the unitary slice $\mathbf{F}_p$ stands in for $\mathbf{U}$, and $\mathbf{P}_p = \mathbf{F}_p\mathbf{F}_p^H$ stands in for $\mathbf{P}_\mathbf{H}$. The numerator matches the sample covariance matrix $\mathbf{Z}\mathbf{Z}^H$ to an estimate of the sample covariance matrix for a p-dimensional source of uncorrelated components, each of common variance. This is only an interpretation, as no such assumption is made in the derivation of the detector.

For the particular case of $p=1$ and $L=2$, the distribution under the null of $\Omega_1 = \lambda_1(\mathbf{Z}\mathbf{Z}^H)/\mathrm{tr}(\mathbf{Z}^H\mathbf{Z})$ has been derived in [8] and is given by

$$f(\Omega_1|\mathcal{H}_0) = c\Omega_1^{N-2}(1-\Omega_1)^{N-2}(2\Omega_1-1)^2, \quad \frac{1}{2} \leq \Omega_1 \leq 1,$$

where $c = \frac{\Gamma(2N)}{\Gamma(N)\Gamma(N-1)}$. For other values of p and L closed-form expressions for the null distribution of Ω_1 are not known.

In the following remark, we prove that for dimension-one signals, the tests derived from first-order and second-order MVN models are equivalent. Thus, the distribution for $L=2$ and first-order models can be used for second-order models.

Remark 2 (The case $p=1$): For this case, the first-order GLR, Ω_1, is

$$\Omega_1 = \frac{\lambda_1(\mathbf{Z}\mathbf{Z}^H)}{\mathrm{tr}(\mathbf{Z}^H\mathbf{Z})}, \qquad \frac{1}{L} < \Omega_1 \leq 1. \tag{32}$$

The GLR Ω_2 in (29) can be written in terms of Ω_1 as

$$\Omega_2 = \frac{1}{L}\left(1-\frac{1}{L}\right)^{L-1}\frac{1}{\Omega_1(1-\Omega_1)^{L-1}}.$$

This transformation is monotonically increasing in the interval $1/L < \Omega_1 \leq 1$, making both GLRs equivalent. Interestingly, this transformation between Ω_1 and Ω_2 coincides with the transformation between the GLRs Λ_1 and Λ_2 on the interval $1/L < \Omega_1 \leq 1$.

V. Simulation Results and Performance Comparisons

The second-order and first-order detectors of Section III have been derived for the case where the subspace $\langle\mathbf{H}\rangle$ is known, which is to say a unitary basis $\mathbf{U}$ is known, but the actual channel map $\mathbf{H}=\mathbf{U}\mathbf{A}$ is unknown. The second-order and first-order detectors of Section IV have been derived for the case where only the dimension of the subspace $\langle\mathbf{H}\rangle$ is known. The question is, how do these detectors, which are summarized in Table I, perform against measurements $\mathbf{Z}=\mathbf{U}\mathbf{A}\mathbf{X}+\mathbf{V}$?

The $p\times N$ signal matrix $\mathbf{X}$ is the signal or symbol matrix. If $\mathbf{X}$ is a matrix of i.i.d. unit variance symbols, then the covariance of the signal matrix is $\mathbf{I}_N\otimes\mathbf{U}\mathbf{A}\mathbf{A}^H\mathbf{U}^H$. If $\mathbf{X}$ is a draw from the Stiefel, then $\mathbf{X}\mathbf{X}^H=\mathbf{I}_p$. We favor these two models, as they model independent sources. In one case each source transmits a sequence of independent unit variance symbols, and in the other case each transmits a unit power sequence. The matrix $\mathbf{A}$ may be written $\mathbf{A}=\mathbf{F}\mathbf{K}\mathbf{G}^H$, where $\mathbf{F}$ is a $p\times p$ unitary matrix, $\mathbf{K}=\mathrm{diag}(k_1,\ldots,k_p)$, and $\mathbf{G}^H$ is a $p\times N$ unitary slice, so that the signal model is $(\mathbf{U}\mathbf{F})\mathbf{K}(\mathbf{G}^H\mathbf{X})$. For normal draws, the distributions of $\mathbf{X}$ and $\mathbf{G}^H\mathbf{X}$ are identical; for Stiefel draws, $\mathbf{X}$ and $\mathbf{G}^H\mathbf{X}$ are indistinguishable as draws from the Grassmannian. So signal measurements are written as $(\mathbf{U}\mathbf{F})\mathbf{K}\mathbf{X}$, with corresponding covariance, if modeled stochastically, $\mathbf{U}\mathbf{F}\mathbf{K}^2\mathbf{F}^H\mathbf{U}^H$. It is as if the signal or symbol matrix $\mathbf{X}$ is poured through the diagonal scaling matrix $\mathbf{K}$, which may also be viewed as the spectrum of the covariance $\mathbf{U}\mathbf{F}\mathbf{K}^2\mathbf{F}^H\mathbf{U}^H$, into the unknown frame $\mathbf{U}\mathbf{F}$ for the subspace $\langle\mathbf{U}\rangle$. This suggests that performance will be determined by the profile of squared eigenvalues $(k_1^2,\ldots,k_p^2)$.

A bulk measure of the eigenvalue spread is given by the spectral flatness, defined as the ratio of the geometric over the arithmetic mean of the eigenvalues k_i^2:

$$\eta = \frac{\left(\prod_{i=1}^{p}k_i^2\right)^{(1/p)}}{\frac{1}{p}\sum_{i=1}^{p}k_i^2}. \tag{33}$$

The spectral flatness parameter in (33) takes values between 0 and 1. For $\eta=1$ all eigenvalues are identical and the spectrum is maximally flat, while lower values of η are associated to spectra with larger spreads. Note that the spectral flatness is invariant to either a constant gain or to a permutation of the k_i^2. Admittedly,

TABLE I
FIRST-ORDER AND SECOND-ORDER DETECTORS FOR KNOWN SUBSPACE AND UNKNOWN SUBSPACE OF KNOWN DIMENSION

	Known subspace	Unknown subspace of known dimension
First-order GLR	$\Lambda_1 = \frac{\mathrm{tr}(\mathbf{Z}^H \mathbf{P}_\mathbf{H} \mathbf{Z})}{\mathrm{tr}(\mathbf{Z}^H \mathbf{Z})}$ (derived in [31])	$\Omega_1 = \frac{\sum_{i=1}^p \lambda_i(\mathbf{Z}\mathbf{Z}^H)}{\mathrm{tr}(\mathbf{Z}^H \mathbf{Z})}$ (derived in [9])
Second-order GLR	$\Lambda_2 = \frac{\mathrm{tr}(\mathbf{S})^L}{\left(\mathrm{tr}(\mathbf{S}) - \sum_{i=1}^p \frac{\hat{\lambda}_i \mu_i}{\lambda_i + 1}\right)^L \prod_{i=1}^p (\hat{\lambda}_i + 1)}$ (derived in this work)	$\Omega_2 = \frac{\left(\frac{\mathrm{tr}(\mathbf{S})}{L}\right)^L}{\left(\prod_{i=1}^p \lambda_i(\mathbf{S})\right)\left(\frac{\sum_{i=p+1}^L \lambda_i(\mathbf{S})}{L-p}\right)^{L-p}}$ (derived in [2])

there is an infinity of profiles that produce the same spectral flatness, and each profile determines its own performance. Spectral flatness is merely a one-parameter characterization of a profile.

In our simulations we use a profile

$$k_i^2 = a^{(i-1)}, \qquad i = 1, \ldots, p, \tag{34}$$

with spectral flatness

$$\eta = \frac{a^{(p-1)/2}(1-a)p}{1-a^p}, \tag{35}$$

where $0 < a < 1$ is a parameter chosen to get the desired spectral flatness. That is, for each choice of the pair (p, η), the value of a is determined from a line search on the interval $0 < a < 1$ for the solution to (35). With this simple one-to-one map from the single parameter a to spectral flatness η, spectral flatness uniquely determines the profile. Of course this is not generally true, as noted above.

The proposed exponential profiles, while certainly not unique to a given spectral flatness, allow us to gain insight about the performances of the detectors based on first- and second-order statistical models. Further, the power delay profile in channel modeling is widely agreed to be an exponentially decreasing function characterized by the delay spread. Why parameterize by k_i^2 and not k_i? Because the k_i^2 model the power spectrum of the source, and it is the power spectrum of the source that is the natural parameterization.

The input and output signal-to-noise ratios (SNR) are defined as follows:

$$\mathrm{SNR}_{in} = 10 \log_{10} \left(\frac{\mathrm{tr}(\mathbf{K}^2)}{L\,\sigma^2} \right)$$

$$\mathrm{SNR}_{out} = \mathrm{SNR}_{in} + 10 \log_{10} N.$$

We use the following simulation setup:

1) For given L, p, N and SNR_{in}, we can wlog set $\sigma^2 = 1$, and solve for the required $\mathrm{tr}(\mathbf{K}^2)$:

$$\sum_{i=1}^p k_i^2 = L\, 10^{\mathrm{SNR}_{in}/10}.$$

2) We determine a in (35) for a given spectral flatness and generate the corresponding profile of eigenvalues $\mathbf{K}^2 = \mathrm{diag}(k_1^2, \ldots, k_p^2)$. These are then scaled to get the desired SNR_{in}.
3) In each realization we randomly generate a unitary basis for the channel $\mathbf{U}$. To this end, we generate an $L \times L$ matrix $\mathbf{B}$ with i.i.d. $\mathcal{CN}(0,1)$ entries, perform its QR decomposition, $\mathbf{B} = \mathbf{QR}$, and then take $\mathbf{U}$ as the first p columns of $\mathbf{Q}$. According to the Bartlett Factorization Theorem [41], $\mathbf{U}$ is drawn uniformly from the Stiefel manifold. Of course, a single draw could have served for all realizations, as the detectors are invariant to left unitary transformations and the noise is white.
4) The signal matrix $\mathbf{X}$ may be a matrix with i.i.d. complex normal entries, a random draw from the Siefel manifold of p-dimensional frames in $\mathbb{C}^N$, or a matrix with i.i.d. draws from discrete constellations. Performance is only weakly dependent on the distribution of the entries of $\mathbf{X}$, provided these entries are i.i.d., as in independent sources, each of which transmits a sequence of i.i.d. symbols. Unless otherwise stated, we chose to generate the signal matrix as $\mathbf{X} \sim \mathcal{CN}_{pN}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{I}_p)$.
5) Finally, the noise is generated as $\mathbf{V} \sim \mathcal{CN}_{LN}(\mathbf{0}, \mathbf{I}_N \otimes \mathbf{I}_L)$ and the observations are $\mathbf{Z} = \mathbf{UFKX} + \mathbf{V}$, with $\mathbf{F}$ a randomly-drawn $p \times p$ unitary matrix unknown to the receiver.
6) For those cases where the subspace is assumed known, the randomly-generated $\mathbf{U}$ is given to the corresponding detectors, Λ_1 and Λ_2. When only the dimension of the subspace is assumed known, then the randomly-generated $\mathbf{U}$ is unknown to the corresponding detectors, Ω_1 and Ω_2.

A. Experiments for the Case of Known Subspace

In this set of experiments the performances of the detectors Λ_2 and Λ_1, derived in Sections III-A and III-B, respectively, are simulated and compared for a variety of parameter choices and signal-to-noise-ratios.

Experiment 1: In the first experiment we consider a sample-rich scenario with $L = 8$ antennas and $N = 64$ snapshots ($N/L = 8$). The input SNR is $\mathrm{SNR}_{in} = -10$ and the output SNR is $\mathrm{SNR}_{out} \approx 8$ dB). Fig. 1 shows the probability of detection vs. the spectral flatness for a fixed $P_{fa} = 10^{-2}$ and different values of p. For dimension-one signals, the GLRs Λ_1 and Λ_2 provide the same performance as we proved in Remark 1. For higher dimension signals and low spectral flatness (or, equivalently, the spread of the signal subspace eigenvalues is large), the test Λ_2 performs better than Λ_1. For spectral flatness values close to one, Λ_1 perfoms better than Λ_2. Interestingly, the performance of the GLR test Λ_1, derived from a first-order model, is almost independent of the spectral flatness.

Experiment 2: In the second experiment, we consider a sample-poor scenario with $L = 32$ antennas and $N = 32$ snapshots, so that $N/L = 1$. The $\mathrm{SNR}_{in} = -14$ ($\mathrm{SNR}_{out} \approx 1$ dB) and the false alarm rate is fixed to $P_{fa} = 10^{-2}$. In the sample-poor regime the question of which detector performs best depends again on the spectral flatness, as shown in Fig. 2. For

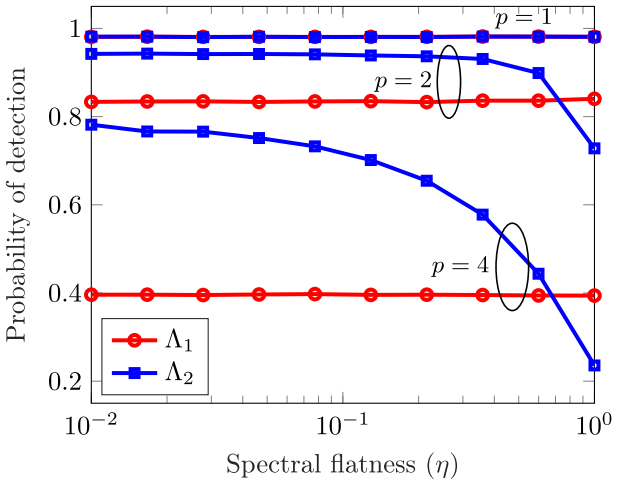

Fig. 1. Large SNR_{out} and rich sample support. Probability of detection vs. η, for fixed $L = 8$, $N = 64$ ($N/L = 8$), $P_{fa} = 10^{-2}$ and $\mathrm{SNR}_{in} = -10$ dB ($\mathrm{SNR}_{out} \approx 8$ dB).

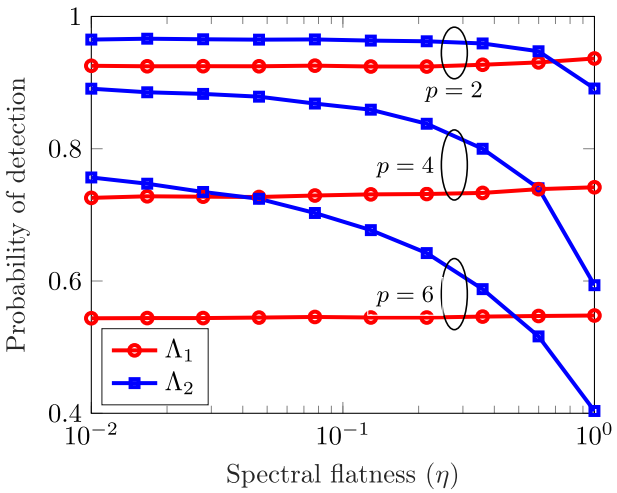

Fig. 2. Low SNR_{out} and poor sample support. Probability of detection vs. η, for fixed $L = 32$, $N = 32$ ($N/L = 1$), $P_{fa} = 10^{-2}$ and $\mathrm{SNR}_{in} = -14$ dB ($\mathrm{SNR}_{out} \approx 1$ dB).

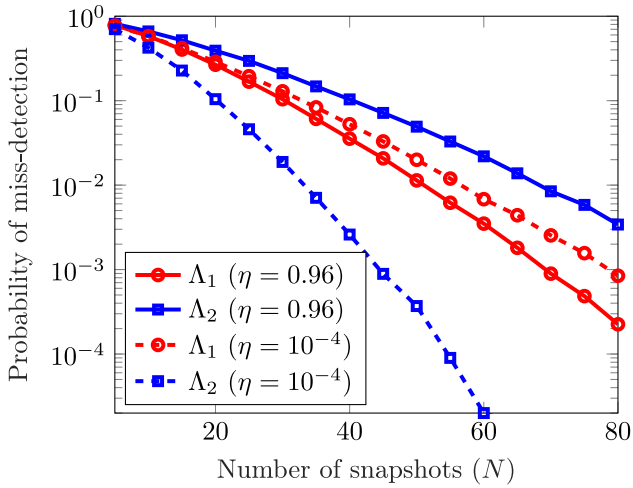

Fig. 3. Moderate SNR_{out} and variable sample support. Probability of miss-detection vs. N, for fixed $L = 20$, $p = 5$, $P_{fa} = 10^{-2}$, $\mathrm{SNR}_{in} = -10$ dB, and different values of η.

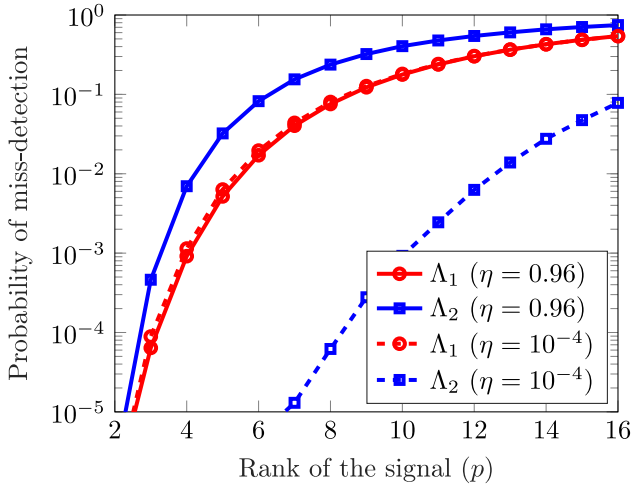

Fig. 4. High SNR_{out} and rich sample support. Probability of miss-detection vs. p, for fixed $L = 32$, $N = 256$, $P_{fa} = 10^{-2}$, $\mathrm{SNR}_{in} = -16$ dB, and different values of η.

sufficiently large eigenvalue spreads, a second-order model outperforms a first-order model.

Experiment 3: In the third experiment, we evaluate the probability of miss-detection vs. the number of snapshots, N. The array has $L = 20$ sensors, the source dimension is $p = 5$, and the input SNR is $\mathrm{SNR}_{in} = -10$ dB. Fig. 3 shows the results for two extreme values of the spectral flatness parameter. When $\eta = 0.96$ (almost flat eigenvalue spectrum), Λ_1 outperforms Λ_2; when $\eta = 10^{-4}$ (eigenvalue spectrum with a large spread), Λ_2 outperforms Λ_1.

Experiment 4: In the fourth experiment, we evaluate the probability of miss-detection vs. the signal dimension p. The array has $L = 32$ antennas and the number of snapshots is $N = 256$ ($N/L = 8$). The input SNR is $\mathrm{SNR}_{in} = -16$ dB and the output SNR is $\mathrm{SNR}_{out} \approx 8$ dB. Fig. 4 shows the results for two extreme values of spectral flatness. When the spectral flatness parameter is close to one, Λ_1 outperforms Λ_2 regardless of source rank. When the spectral flatness parameter is $\eta = 10^{-4}$, Λ_2 outperforms Λ_1 regardless of source rank.

B. Experiments for the Case of Unknown Subspace of Known Dimension

Here we consider the case where the subspace is unknown but its dimension is known. To simplify the comparison between the cases of known subspace and unknown subspace of known dimension, we conduct experiments with similar parameters as those used in Section V-A.

Experiment 5: In the fifth experiment we evaluate the probability of detection vs. spectral flatness. The array has $L = 8$ antennas, and $N = 64$ snapshots. The input SNR is $\mathrm{SNR}_{in} = -8$ dB and the output SNR is $\mathrm{SNR}_{out} \approx 10$ dB. Fig. 5 shows the probability of detection vs. the spectral flatness. When the source dimension is 1, Ω_1 and Ω_2 have identical performances, as proved in Remark 2. When the source dimension exceeds

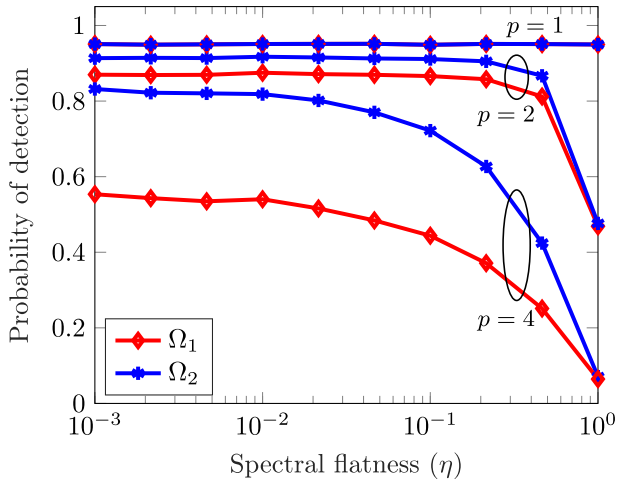

Fig. 5. High SNR_{out} and rich sample support. Probability of detection vs. η, for fixed $L = 8$, $N = 64$ ($N/L = 8$), $P_{fa} = 10^{-2}$ and $\text{SNR}_{in} = -8$ dB ($\text{SNR}_{out} \approx 10$ dB).

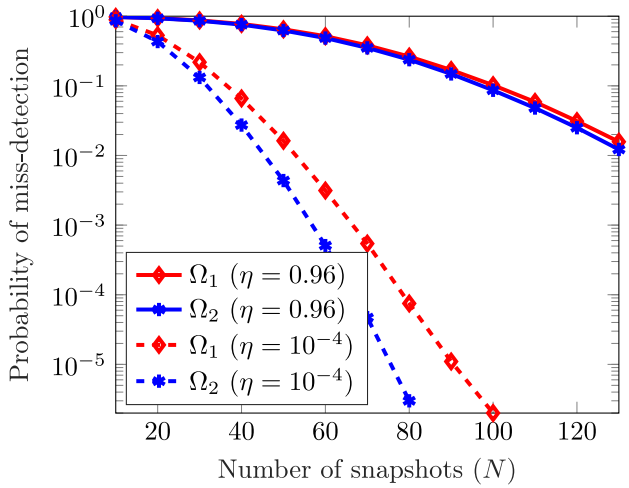

Fig. 7. Variable SNR_{out} and variable sample support. Probability of miss-detection vs. N, for fixed $L = 20$, $p = 5$, $P_{fa} = 10^{-2}$, $\text{SNR}_{in} = -8$ dB, and different values of η.

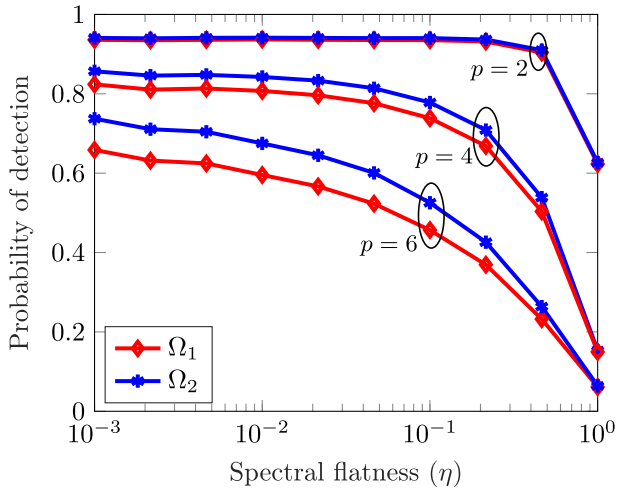

Fig. 6. Moderate SNR_{out} and poor sample support. Probability of detection vs. η, for fixed $L = 32$, $N = 32$ ($N/L = 1$), $P_{fa} = 10^{-2}$ and $\text{SNR}_{in} = -10$ dB ($\text{SNR}_{out} \approx 5$ dB).

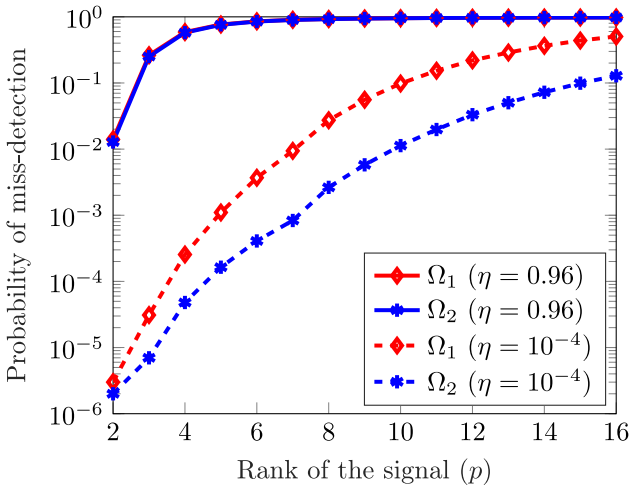

Fig. 8. High SNR_{out} and rich sample support. Probability of miss-detection vs. p, for fixed $L = 32$, $N = 256$, $P_{fa} = 10^{-2}$, $\text{SNR}_{in} = -14$ dB, and different values of η.

1, Ω_2 provides better perfomance than Ω_1 regardless of the spectral flatness. The performance of both tests degrades when the spectral flatness is close to one.

Experiment 6: In this experiment, we consider a sample-poor scenario with $L = 32$ antennas, $N = 32$ snapshots ($N/L = 1$). The input SNR is $\text{SNR}_{in} = -10$ dB and the output SNR is $\text{SNR}_{out} = 5$ dB. The probability of detection vs. η for different source dimensions is depicted in Fig. 6 . For all values of spectral flatness, the GLR Ω_2 is slightly better than the GLR Ω_1, regardless of the source dimension. The difference in perfomance between them is not as significant as in the sample-rich scenario of Fig. 5.

Experiment 7: In this experiment, we consider a scenario with $L = 20$ sensors and $p = 5$ sources. The input SNR is $\text{SNR}_{in} = -8$ dB. The number of snapshots is variable. Fig. 7 shows the miss-detection probability vs. the number of snapshots. The second-order GLR Ω_2 outperforms the first-order GLR Ω_1. The improvement is small when the spectral flatness is close to one, but it becomes more significant when the spectral flatness is $\eta = 10^{-4}$.

Experiment 8: In this experiment, we evaluate the probability of miss-detection vs. the signal dimension p. The array has $L = 32$ antennas and the number of snapshots is $N = 256$ ($N/L = 8$). The input SNR is $\text{SNR}_{in} = -14$ dB and the output SNR is $\text{SNR}_{out} \approx 10$ dB. Fig. 8 shows the results for two extreme values of spectral flatness. The GLR Ω_2 outperforms Ω_1 regardless of the source dimension and the spectral flatness. The benefit of Ω_2 over Ω_1 is insignificant for $\eta = 0.96$, but significant for $\eta = 10^{-4}$.

C. Known Subspace Detectors (Λ_2, Λ_1) Vs. Unknown Subspace Detectors (Ω_2, Ω_1)

Experiment 9: In this experiment, we analyze the value of knowing the subspace in which the source transmissions lie

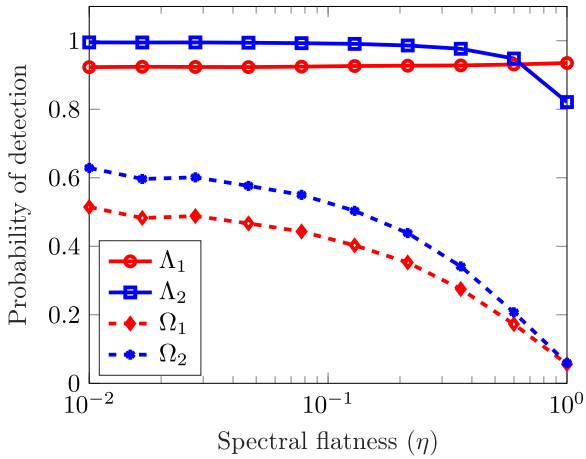

Fig. 9. Large SNR_{out} and rich sample support. Probability of detection vs. η, for fixed $L = 16$, $N = 64$ ($N/L = 4$), $p = 4$, $P_{fa} = 10^{-2}$ and $\mathrm{SNR}_{in} = -11$ dB ($\mathrm{SNR}_{out} \approx 7$ dB).

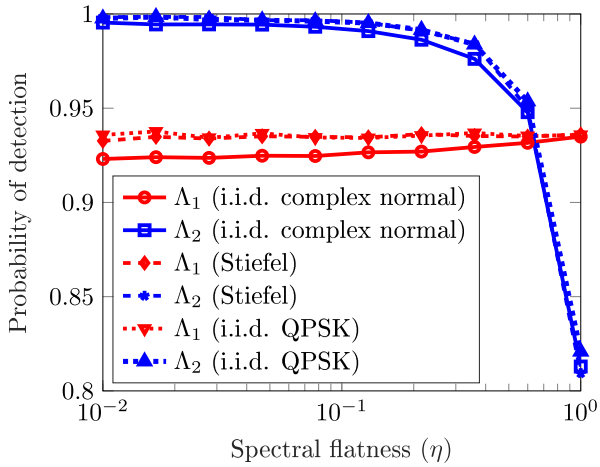

Fig. 10. Performance of detectors Λ_1 and Λ_2 for different distribution of the source signal $\mathbf{X}$. The scenario has fixed $L = 16$, $N = 64$ ($N/L = 4$), $p = 4$, $P_{fa} = 10^{-2}$ and $\mathrm{SNR}_{in} = -11$ dB ($\mathrm{SNR}_{out} \approx 7$ dB).

versus knowing only its dimension. To this end, we compare the performance of the detectors Λ_i (known subspace) vs. the detectors Ω_i (unknown subspace of known dimension). The array has $L = 16$ sensors, the number of snapshots is $N = 64$ ($N/L = 4$) and the number of sources is $p = 4$. The input SNR is $\mathrm{SNR}_{in} = -11$ dB and the output SNR is $\mathrm{SNR}_{out} \approx 7$ dB. Fig. 9 shows the probability of detection vs. the spectral flatness for a fixed $P_{fa} = 10^{-2}$. In this simulation the assumed subspace for Λ_2 and Λ_1 is matched to the actual signal subspace, so this simulation demonstrates the value of exploiting subspace knowledge when it is available. We do not present simulation results for the case where the assumed subspace is mismatched to the actual signal subspace, as in this case the performance of the known-subspace detectors can be arbitrarily bad. So the detector that assumes only known dimension is always the robust alternative if subspace knowledge is not trustworthy.

D. Performance Comparison for Different Distributions of $\mathbf{X}$

Experiment 10: So far all the experiments have been performed with source signals generated as random matrices of i.i.d. complex normals. This experiment demonstrates the very weak dependence of detector perfomance on the distribution of the signal matrix $\mathbf{X}$. To this end, we consider a scenario with $L = 16$ antennas, $p = 4$ sources and $N = 64$ snapshots. The 4×64 source matrix, $\mathbf{X}$, is generated according to one of the following distributions:

- A random matrix of i.i.d. complex normals, $\mathcal{CN}(0, 1)$.
- A draw from the Stiefel manifold of p-dimensional frames in $\mathbb{C}^N$. The draws are uniform with respect to the Haar measure on the manifold.
- A random matrix of i.i.d. unit-norm quadrature phase shift keyed (QPSK) symbols.

Fig. 10 shows the probability of detection vs. the spectral flatness for a fixed $P_{fa} = 10^{-2}$ when the subspace is known (detectors Λ_1 and Λ_2), while Fig. 11 shows the results when only the dimension of the subspace is known (detectors Ω_1 and Ω_2). For all four detectors, the performance depends only

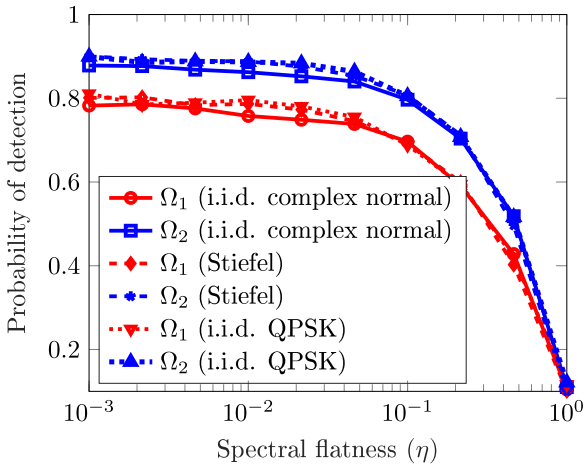

Fig. 11. Performance of detectors Ω_1 and Ω_2 for different distribution of the source signal $\mathbf{X}$. The scenario has fixed $L = 16$, $N = 64$ ($N/L = 4$), $p = 4$, $P_{fa} = 10^{-2}$ and $\mathrm{SNR}_{in} = -10$ dB ($\mathrm{SNR}_{out} \approx 8$ dB).

weakly on the univariate distribution of the elements of $\mathbf{X}$. One explanation for this behavior could be that all detectors use second-order statistics (in fact, eigenvalues) in their designs and implementations. And these second-order statistics are relatively invariant to the univariate distribution of elements of $\mathbf{X}$.

VI. Conclusion

In this paper we have studied the problem of detecting a subspace signal from multiple snapshots in a multi-sensor array. Each snapshot consists of a subspace signal plus additive Gaussian noise. The covariance of the noise is known, up to unknown scale. There are two models for the subspace signal: the subspace may be known or only its dimension may be known. To say the subspace is known is to say a basis $\mathbf{U}$ for the subspace $\langle \mathbf{H} \rangle$ is known, but not the actual channel map $\mathbf{H} = \mathbf{UA}$. To say only the subspace dimension is known is to say that only the dimension of the subspace $\langle \mathbf{H} \rangle$ is known.

There are two statistical models for the measurements: in a first-order model, the subspace signal appears in the mean-value vector of a multivariate normal measurement and the additive noise variance determines its covariance matrix; in a second-order model, the covariance of the subspace signal adds to the covariance of the additive noise to make the covariance matrix of a multivariate normal measurement. Thus, there are four variations on the detection problem: known subspace and first-order statistical model; known subspace and second-order statistical model; known subspace dimension (only) and first-order statistical model; and known subspace dimension (only) and second-order statistical model. For three of these models the GLR detectors are known, and they have been reported in the literature. But for a known subspace and a second-order model for the measurements, the GLR is derived, we believe for the first time, in this paper. This is our first original result.

All four detectors are scale-invariant, as we insist they must be when the additive noise variance is unknown. It is shown that for a dimension-1 source radiating or transmitting to a multi-sensor array, the GLR based on a first-order statistical model is equivalent to the GLR based on a second-order statistical model. That is, each is a monotone function of the other, and there can be no difference in performance. This equivalence holds for the case where the subspace is known and for the case where only the subspace dimension is known. This is our second original result.

Our experimental findings are based on a one-parameter spectral profile that is determined by the one-parameter spectral flatness. When the channel subspace is known, the performance of the GLR based on a second-order statistical model depends significantly on the spread of the signal subspace eigenvalues. The performance of the detector based on a first-order model, however, does not depend much on the eigenvalue spread. For large eigenvalue spreads, the GLR based on a second order statistical model outperforms the GLR based on a first-order statistical model. For small eigenvalue spreads, the GLR based on a first-order statistical model outperforms the GLR based on a second-order model.

For the case where only the subspace dimension is known, the GLR based on a second-order statistical model outperforms the GLR based on a first-order statistical model, regardless of the eigenvalue spread, and for all choices of the parameters (L, p, N, SNR). This claim is consistent with the findings of [9], based on their experiments. Depending upon the eigenvalue spread and the ratio of sensors to sources the difference in performance may be significant or it may be negligible.

There remains the question of model mismatch. In first-order models, no assumption is made about the spacetime signal sequence. It may be stationary in time or transient. In the second-order model it is assumed stationary. So if the signal sequence is transient, it is expected that the performance of the second-order detector will degrade, whereas the first-order detector will not. This issue remains to be explored.

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