A GENERALIZATION OF THE MAGNITUDE SQUARED COHERENCE SPECTRUM FOR MORE THAN TWO SIGNALS: DEFINITION, PROPERTIES AND ESTIMATION

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ABSTRACT

The coherence spectrum is a well-known measure of the linear statistical relationship between two time series. In this paper, we extend this concept to several processes and define the generalized magnitude squared coherence (GMSC) spectrum as a function of the largest eigenvalue of a matrix containing all the pairwise complex coherence spectra. The GMSC is bounded between zero and one, and attains its maximum when all the processes are perfectly correlated at a given frequency. Furthermore, three different GMSC spectrum estimators, extending those previously proposed for the MSC of two processes, are presented. Specifically, we compare the Welch method, the minimum variance distortionless response (MVDR) estimator and a new estimator based on canonical correlation analysis (CCA).

***Index Terms*—** Generalized magnitude squared coherence (GMSC) spectrum, filter-bank approach, canonical correlation analysis (CCA), minimum variance distortionless response (MVDR) filter.

1. INTRODUCTION

The magnitude squared coherence spectrum (MSC) provides a frequency-dependent measure of the linear relationship between two stationary random processes, which can also be interpreted as a correlation coefficient in the frequency domain [1]. For gaussian processes it also provides a measure of the mutual information [2]. Despite its usefulness, when more than two signal are involved a commonly accepted generalization of the MSC does not exist yet and measuring all the pairwise MSC spectra is not practical. For instance, for eight random processes there are twenty-eight different MSC spectra.

In an attempt to fill this gap, in this paper we present a possible generalization of the MSC for several stationary processes. The generalized MSC (GMSC) is defined as a function of the largest eigenvalue of a matrix containing the pairwise complex coherence spectra. Some of its properties are also described in the paper: for instance, the GMSC is bounded between zero and one and it allows us to quantify the contribution of each process to the overall correlation. Additionally, for two processes the GMSC obviously reduces to the conventional definition.

Finally, we present and compare three techniques for estimating the GMSC. Two of them are straightforward extensions of the Welch method [3], and the minimum variance distortionless response (MVDR) approach [4] used to estimate the conventional MSC. The third one is based on a generalization of canonical correlation analysis (CCA) to several data sets [5], and also extends a recently proposed MSC estimator [6]. Some simulation examples show that the CCA-based estimator provides better resolution than the Welch's method and also avoids the signal mismatch problem associated to the MVDR estimator.

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2. DEFINITION OF THE MAGNITUDE SQUARED COHERENCE SPECTRUM FOR MULTIPLE SIGNALS

In this section we propose a generalization of the magnitude squared coherence spectrum (MSC) for $M \geq 2$ signals and discuss some of its properties. Let us consider M zero-mean stationary complex time series $x_1[n], \ldots, x_M[n]$; and define the complex coherence spectrum [1] between the *i*-th and *j*-th signals as

$$C_{x_i x_j}(\omega) = \frac{S_{x_i x_j}(\omega)}{\sqrt{S_{x_i x_i}(\omega) S_{x_j x_j}(\omega)}}, \quad \forall i, j = 1, \cdots, M,$$

where $S_{x_i x_j}(\omega)$ is the cross-spectrum and $S_{x_i x_i}(\omega)$ is the power spectral density of the *i*-th signal. Interestingly, the complex coherence spectrum can also be seen as the cross-spectrum of the prewhitened signals.

In the case of $M = 2$ time series, the MSC is defined as $\gamma^2(\omega) = |C_{x_1 x_2}(\omega)|^2$ [1]. In order to extend this idea to the general case of $M \geq 2$ stationary random processes we define the matrix $\mathbf{\Sigma}_x(\omega) \in \mathbb{C}^{M \times M}$ containing all the pairwise complex coherence spectra as

$$\mathbf{\Sigma}_x(\omega) = \begin{bmatrix} 1 & C_{x_1 x_2}(\omega) & \ldots & C_{x_1 x_M}(\omega) \\ C_{x_2 x_1}(\omega) & 1 & \ldots & C_{x_2 x_M}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ C_{x_M x_1}(\omega) & C_{x_M x_2}(\omega) & \ldots & 1 \end{bmatrix},$$

which can be rewritten as

$$\mathbf{\Sigma}_x(\omega) = \mathbf{D}_x^{-1/2}(\omega) \mathbf{S}_x(\omega) \mathbf{D}_x^{-1/2}(\omega), \tag{1}$$

where

$$\mathbf{S}_x(\omega) = \begin{bmatrix} S_{x_1 x_1}(\omega) & S_{x_1 x_2}(\omega) & \ldots & S_{x_1 x_M}(\omega) \\ S_{x_2 x_1}(\omega) & S_{x_2 x_2}(\omega) & \ldots & S_{x_2 x_M}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ S_{x_M x_1}(\omega) & S_{x_M x_2}(\omega) & \ldots & S_{x_M x_M}(\omega) \end{bmatrix},$$

and $\mathbf{D}_x(\omega)$ is a diagonal matrix whose entries are $[\mathbf{D}_x(\omega)]_{i,i} = S_{x_i x_i}(\omega)$.

Definition: *The generalized magnitude squared coherence spectrum (GMSC) is defined as* $\gamma^2(\omega)$*, where*

$$\gamma(\omega) = \frac{1}{M-1} \left(\lambda_{MAX}(\mathbf{\Sigma}_x(\omega)) - 1 \right),$$

and $\lambda_{MAX}(\mathbf{\Sigma}_x(\omega))$ *is the largest eigenvalue of the matrix* $\mathbf{\Sigma}_x(\omega)$*.*

From (1), it is easy to prove that $\lambda_{MAX}(\mathbf{\Sigma}_x(\omega))$ is also the largest eigenvalue of the following generalized eigenvalue (GEV) problem

$$\mathbf{S}_x(\omega)\tilde{\mathbf{v}}(\omega) = \lambda(\omega)\mathbf{D}_x(\omega)\tilde{\mathbf{v}}(\omega), \tag{2}$$

where $\tilde{\mathbf{v}}(\omega) = \mathbf{D}_x^{-1/2}(\omega)\mathbf{v}(\omega)$ is the generalized eigenvector and $\mathbf{v}(\omega)$ is the eigenvector of $\mathbf{\Sigma}_x(\omega)$. Now, we present some properties of the GMSC.

Property 1: *The GMSC spectrum is bounded between* 0 *and* 1*, i.e.,*

$$0 \leq \gamma^2(\omega) \leq 1.$$

Proof. Taking into account that the trace of $\mathbf{\Sigma}_x(\omega)$ is

$$\text{Tr}\left(\mathbf{\Sigma}_x(\omega)\right) = \sum_{i=1}^{M} \lambda^{(i)}(\omega) = M,$$

where $\lambda^{(i)}(\omega) \geq 0$ are the eigenvalues, it is clear that the maximum value of $\lambda^{(1)}(\omega)$ is M when $\lambda^{(i)}(\omega) = 0$, $i = 2, \ldots, M$, whereas its minimum value is 1 when all the eigenvalues are equal $\lambda^{(i)}(\omega) = 1$, $i = 1, \ldots, M$. Therefore, $1 \leq \lambda_{MAX}(\mathbf{\Sigma}_x(\omega)) \leq M$, which implies $0 \leq \gamma^2(\omega) \leq 1$. □

Property 2: *The GMSC spectrum is maximum when the* M *time series are perfectly pairwise correlated at that frequency, and minimum when all the signals are uncorrelated.*

Proof. In the case of perfectly pairwise correlated signals at frequency ω the matrix $\mathbf{\Sigma}_x(\omega)$ becomes an all-ones matrix, and its largest eigenvalue is $\lambda_{MAX}(\mathbf{\Sigma}_x(\omega)) = M$. On the other hand, in the case of uncorrelated signals we have $\mathbf{\Sigma}_x(\omega) = \mathbf{I}$ and $\lambda_{MAX}(\mathbf{\Sigma}_x(\omega)) = 1$, which concludes the proof. □

Property 3: *In the case of* $M = 2$ *signals, the GMSC spectrum reduces to the standard MSC spectrum definition.*

Proof. For $M = 2$, it is easy to see that $\mathbf{\Sigma}_x(\omega)$ becomes

$$\mathbf{\Sigma}_x(\omega) = \begin{bmatrix} 1 & C_{x_1x_2}(\omega) \\ C^*_{x_1x_2}(\omega) & 1 \end{bmatrix},$$

and its largest eigenvalue is $\lambda_{MAX}(\mathbf{\Sigma}_x(\omega)) = 1 + |C_{x_1x_2}(\omega)|$, which yields $\gamma^2(\omega) = (\lambda_{MAX} - 1)^2 = |C_{x_1x_2}(\omega)|^2$. □

Interestingly, the *i*-th coefficient $v_i(\omega)$ of the eigenvector $\mathbf{v}(\omega)$ associated to the largest eigenvalue of $\mathbf{\Sigma}_x(\omega)$ measures the contribution of the *i*-th signal to the GMSC at frequency ω. For instance, if there are M' ($M \geq M' > 1$) signals perfectly pairwise correlated at frequency ω and the remaining $M - M'$ signals are uncorrelated, then the squared modulus of the components of the eigenvector will be $|v_i(\omega)|^2 = 1/M'$ for the perfectly correlated signals and 0 for the uncorrelated ones. Let us consider the following example, there are $M = 3$ random processes, $M' = 2$ signals are perfectly correlated at frequency ω and the third one is uncorrelated with both. In this example the matrix $\mathbf{\Sigma}_x(\omega)$ is

$$\mathbf{\Sigma}_x(\omega) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

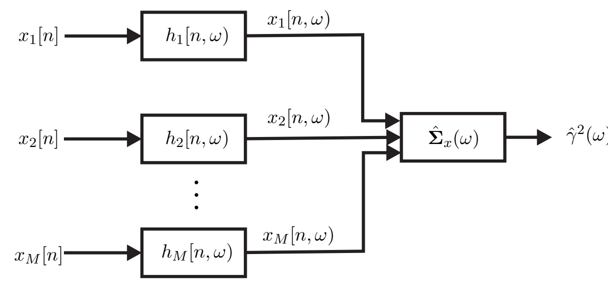

Fig. 1. Estimation of the GMSC based on a filter-bank approach.

the GMSC at frequency ω is $\gamma^2(\omega) = 1/4$ and the corresponding eigenvector is $\mathbf{v}(\omega) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}^T$.

3. ESTIMATION OF THE GMSC BASED ON A FILTER-BANK APPROACH

In this section we propose a straightforward extension of the techniques proposed in [3, 4] for the estimation of the magnitude squared coherence spectrum for $M = 2$ signals. These techniques are based on a filter-bank interpretation of the cross-spectra and the power spectral densities.

The proposed filter-bank approach for GMSC estimation is shown in Fig. 1. The input signals are filtered by a set of M bandpass filters $h_i[n,\omega)$ centered at the frequency ω and the matrix $\mathbf{\Sigma}_x(\omega)$ is estimated from a finite number (N) of observations of the output signals $x_i[n,\omega)$. Specifically, the filter outputs are

$$x_i[n,\omega) = x_i[n] * h_i[n,\omega) \qquad i = 1, \ldots M,$$

and they are normalized as

$$\tilde{x}_i[n,\omega) = \frac{x_i[n,\omega)}{\sqrt{\sum_{n=0}^{N-1} |x_i[n,\omega)|^2}}.$$

Thus, the matrix $\mathbf{\Sigma}_x(\omega)$ can be estimated as

$$\hat{\mathbf{\Sigma}}_x(\omega) = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{\mathbf{x}}[n,\omega)\tilde{\mathbf{x}}[n,\omega)^H,$$

where $\tilde{\mathbf{x}}[n,\omega) = [\tilde{x}_1[n,\omega), \ldots, \tilde{x}_M[n,\omega)]^T$. Finally, the estimate $\hat{\gamma}^2(\omega)$ of the GMSC spectrum is directly obtained from the largest eigenvalue of $\hat{\mathbf{\Sigma}}_x(\omega)$.

Regarding the bandpass filters, we present two different alternatives.

- The well-known Welch's technique [3] defines a set of bandpass filters given by $h_i^{(\text{Welch})}[n,\omega) = w[n]e^{j\omega n}$, where $w[n]$ is a window of size L.
- The direct application of the MVDR criterion [4] provides a set of filters $\mathbf{h}_i^{(\text{MVDR})}(\omega) = \hat{\mathbf{R}}^{-1}_{\mathbf{x}_i\mathbf{x}_i}\mathbf{f}(\omega)/\mathbf{f}^T(\omega)\hat{\mathbf{R}}^{-1}_{\mathbf{x}_i\mathbf{x}_i}\mathbf{f}^*(\omega)$, where $\mathbf{f}(\omega) = 1/\sqrt{L}[1, e^{j\omega}, \ldots, e^{j\omega(L-1)}]^T$ is the Fourier vector of length L and $\hat{\mathbf{R}}_{\mathbf{x}_i\mathbf{x}_i}$ is an estimate of the correlation matrix of $x_i[n]$.

4. ESTIMATION OF THE GMSC SPECTRUM BASED ON CCA

Let us start by considering the asymptotic case when $L \to \infty$, and writing

$$S_{x_i x_j}(\omega) = \mathbf{f}^H(\omega)\mathbf{R}_{x_i x_j}\mathbf{f}(\omega),$$

where $\mathbf{R}_{x_i x_j}$ is the infinite Toeplitz cross-correlation matrix between the i-th and j-th signals, and $\mathbf{f}(\omega)$ is the Fourier vector of infinite length at frequency ω. With this definition, the matrices $\mathbf{S}_x(\omega)$ and $\mathbf{D}_x(\omega)$ can be rewritten as

$$\mathbf{S}_x(\omega) = \mathbf{F}^H(\omega) \overbrace{\begin{bmatrix} \mathbf{R}_{x_1x_1} & \dots & \mathbf{R}_{x_1x_M} \\ \vdots & \ddots & \vdots \\ \mathbf{R}_{x_Mx_1} & \dots & \mathbf{R}_{x_Mx_M} \end{bmatrix}}^{\mathbf{R}} \mathbf{F}(\omega), \quad (3)$$

$$\mathbf{D}_x(\omega) = \mathbf{F}^H(\omega) \underbrace{\begin{bmatrix} \mathbf{R}_{x_1x_1} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{R}_{x_Mx_M} \end{bmatrix}}_{\mathbf{D}} \mathbf{F}(\omega), \quad (4)$$

where

$$\mathbf{F}(\omega) = \begin{bmatrix} \mathbf{f}(\omega) & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{f}(\omega) \end{bmatrix}.$$

Thus, taking into account that the matrices $\mathbf{R}_{x_i x_j}$ are diagonalized by the Fourier vectors, the eigenvalue (EV) problem $\mathbf{\Sigma}_x(\omega)\mathbf{v}(\omega) = \lambda(\omega)\mathbf{v}(\omega)$ can be rewritten as

$$\mathbf{D}^{-1/2}\mathbf{R}\mathbf{D}^{-1/2}\mathbf{w}(\omega) = \lambda(\omega)\mathbf{w}(\omega), \quad (5)$$

where $\mathbf{w}(\omega) = \mathbf{F}(\omega)\mathbf{v}(\omega)$, or equivalently

$$\mathbf{w}(\omega) = \left[\mathbf{w}_1^T(\omega), \dots, \mathbf{w}_M^T(\omega)\right]^T,$$

with $\mathbf{w}_k(\omega) = v_k(\omega)\mathbf{f}(\omega)$.

Interestingly, the EV problem in (5) can be seen as the classical formulation of the maximum variance (MAXVAR) canonical correlation analysis (CCA) technique [5]. Analogously to (2), the CCA-MAXVAR problem can be rewritten as [7]

$$\mathbf{R}\tilde{\mathbf{w}}(\omega) = \lambda(\omega)\mathbf{D}\tilde{\mathbf{w}}(\omega),$$

where $\tilde{\mathbf{w}}(\omega) = \mathbf{D}^{-1/2}\mathbf{w}(\omega)$.

From (5) it is clear that, in the asymptotic case $L \to \infty$, the GMSC spectrum can be directly obtained from the eigenvalues of $\mathbf{D}^{-1/2}\mathbf{R}\mathbf{D}^{-1/2}$. However, in a practical situation L is finite, and the cross-correlation matrices are estimated from a limited number of observations, which translates into a difference between the ideal eigenvectors $\mathbf{w}(\omega) = \mathbf{F}(\omega)\mathbf{v}(\omega)$ and the actual ones $\hat{\mathbf{w}}^{(p)} = \left[\hat{\mathbf{w}}_1^{(p)T}, \dots, \hat{\mathbf{w}}_M^{(p)T}\right]^T$ of

$$\hat{\mathbf{D}}^{-1/2}\hat{\mathbf{R}}\hat{\mathbf{D}}^{-1/2}\hat{\mathbf{w}}^{(p)} = \hat{\lambda}^{(p)}\hat{\mathbf{w}}^{(p)}, \quad (6)$$

where $\hat{\mathbf{D}} \in \mathbb{C}^{LM \times LM}$ and $\hat{\mathbf{R}} \in \mathbb{C}^{LM \times LM}$ are the estimated finite size versions of $\mathbf{D}$ and $\mathbf{R}$.

In order to obtain an accurate GMSC estimate from the solutions of (6), we propose a method based on a reduced-rank representation of the matrix $\hat{\mathbf{D}}^{-1/2}\hat{\mathbf{R}}\hat{\mathbf{D}}^{-1/2}$, which generalizes the technique presented in [6] for the estimation of the conventional MSC. Specifically, the proposed GMSC estimate is obtained as

$$\hat{\gamma}(\omega) = \frac{1}{M-1}\sum_{p=1}^{P}\sum_{k=1}^{M}(\hat{\lambda}^{(p)} - 1)\left|\mathbf{f}^H(\omega)\hat{\mathbf{w}}_k^{(p)}\right|^2,$$

where P is the selected rank, and $\hat{\lambda}^{(p)}$, $p = 1, \dots, P$, are the P largest eigenvalues of (6). Finally, it is easy to prove that, in the asymptotic case where $L, P, N \to \infty$, the proposed estimation technique yields

$$\begin{aligned}\hat{\gamma}(\omega) &= \frac{1}{M-1}\int_{\omega'}\sum_{k=1}^{M}\left(\lambda(\omega') - 1\right)\left|\mathbf{f}^H(\omega)\mathbf{f}(\omega')v_k(\omega')\right|^2 d\omega' \\ &= \frac{1}{M-1}\int_{\omega'}\left(\lambda(\omega') - 1\right)\delta(\omega - \omega')d\omega' = \gamma(\omega),\end{aligned}$$

which coincides with the GMSC definition.

5. SIMULATION RESULTS

In this section we evaluate the performance of the proposed methods for the estimation of the GMSC spectrum by means of two different examples. In all the simulations, we have considered $N = 1024$ observations of $M = 3$ different signals. We have selected $L = 100$ and a Hanning window for the Welch approach. The GMSC spectrum has been evaluated at $K = 200$ equispaced frequencies. Finally, in both cases, the order of the CCA rank-reduction technique is $P = 10$.

In the first example we have considered the following signals

$$x_k[n] = w_k[n] + \sum_{i=1}^{N_f}\cos\left(2\pi\nu_k^{(i)}n + \phi_k\right), \quad k = 1, \dots, 3,$$

where $w_k[n]$ are independent zero-mean and real Gaussian random processes with unit variance, and the phases ϕ_k are uniform random variables between 0 and 2π. Finally, we have considered the set of frequencies shown in Table 1 ($N_f = 5$). Here, we must point out that all frequencies except $\nu_k^{(3)}$ coincide with frequencies of the Fourier grid. The GMSC estimates are shown in Figs. 2.a) - 2.c),

First Example	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
$\nu_1^{(i)}$	0.05	0.06	0.152	0.20	0.25
$\nu_2^{(i)}$	0.05	0.06	0.152	0.20	0.35
$\nu_3^{(i)}$	0.05	0.06	0.152	0.25	0.4

Table 1. Frequencies of the harmonic random processes of the first example.

where we can see that the best results are obtained by the proposed CCA-based technique, which eliminates the spurious correlations; and by the MVDR based approach, which provides the highest spectral resolution. However, the MVDR estimate is severely degraded for the frequency $\nu_k^{(3)}$,which can be seen as the equivalent to the well-known signal mismatch problem [6]. As expected, the maximum of the GMSC is at the frequencies when all the signals are perfectly correlated.

In the final example, the three signals are generated as

$$x_k[n] = s[n] + w_k[n], \quad k = 1, \dots, 3,$$

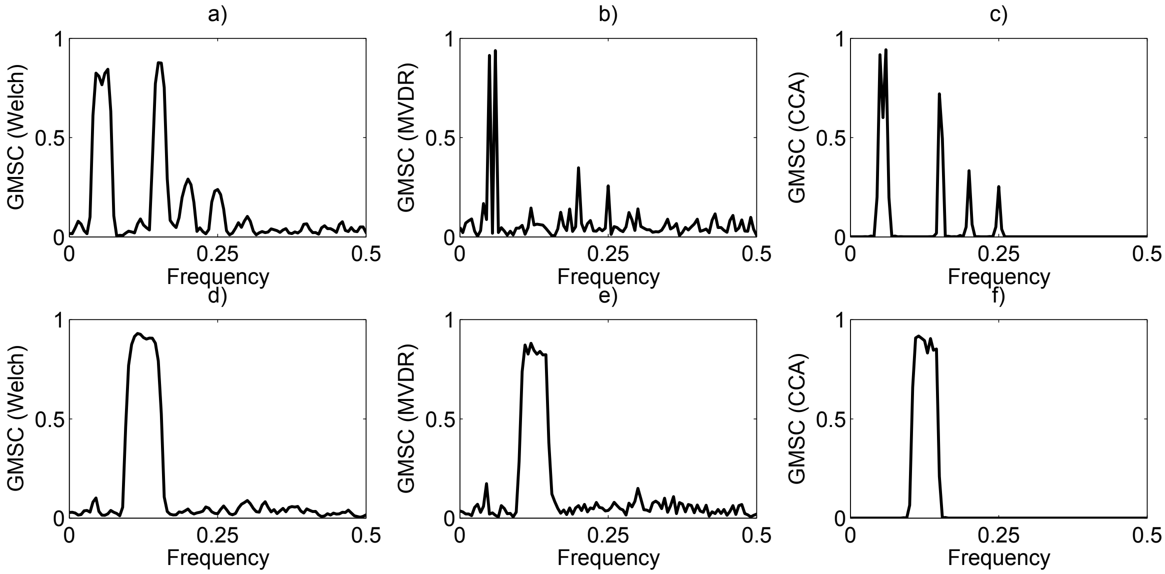

Fig. 2. GMSC spectrum estimates. Subfigures a)-c) correspond to the first example and subfigures d)-f) correspond to the second example.

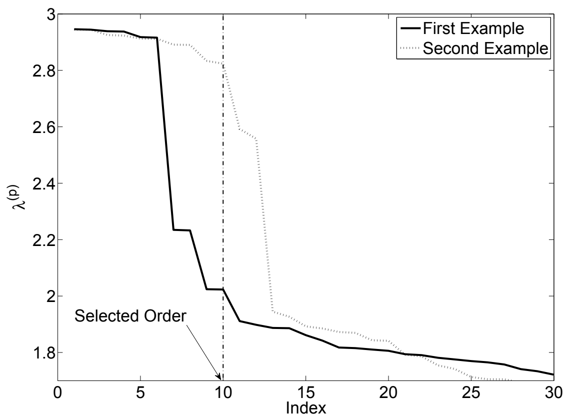

Fig. 3. Main eigenvalues for the two examples presented.

where $w_k[n]$ are independent zero-mean and real Gaussian random processes with unit variance, and the common signal $s[n]$ is a narrowband zero-mean real Gaussian process with unit power and passband between 0.1 and 0.15. The results are shown in Figs. 2.d) - 2.f), where we can see that the reduced-rank CCA approach provides the most accurate estimate.

Finally, Fig. 3 shows the eigenvalues obtained in the previous examples. In both cases the number of dominant eigenvalues is approximately $P = 10$, which justifies our election of the order for the reduced-rank CCA technique. However, in a realistic scenario the order of the reduced-rank technique must be estimated. This problem will be addressed in future work.

6. CONCLUSIONS

In this paper the magnitude squared coherence (MSC) spectrum has been generalized to the case of multiple signals. The proposed generalization is based on the largest eigenvalue of a matrix containing all the pairwise coherence spectra and, in the case of two signals, it reduces to the classical MSC formulation. Additionally, we have proposed a technique for the estimation of the generalized MSC (GMSC), which can be reformulated as a generalized canonical correlation analysis (CCA) problem. Unlike other well-known approaches, such as the Welch and minimum variance distortionless response (MVDR), the proposed estimation technique provides a high resolution and is not affected by signal mismatch problems, which has been illustrated by means of some simulation examples.

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