Passive Localization of a Radiating Gaussian Subspace Signal

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***Abstract*—This work investigates the problem of passive source localization (PSL) using observations from two distributed sensors. The transmitted signal is assumed to lie in a known low-dimensional subspace, with its location in this subspace determined by a colored Gaussian random vector. This second-order model contrasts with the first-order model of [1], which assigns no distribution to the location of the signal in the known subspace. We derive the generalized likelihood ratio for this second-order model and compare its performance to the first-order model of [1] and the second-order model of [2], where the location vector is assumed to be a white Gaussian vector. Numerical experiments reveal that the proposed detector achieves superior performance for highly colored signals, as quantified by the spectral flatness of the eigenvalues of the signal covariance matrix. Performance degrades as the signal becomes increasingly white, aligning with theoretical expectations.**

***Index Terms*—Alternating optimization (AO), generalized likelihood ratio (GLR), maximum likelihood (ML) estimation, minorization-maximization (MM) algorithms, passive multichannel detection, passive source localization.**

I. Introduction

This work addresses the problem of passively detecting a radiating source using observations from two distributed sensors [3]–[7]. The transmitted signal is assumed to lie in a known low-dimensional subspace; however, the channel and carrier phase are unknown. As a result, the channels between the transmitter and the sensors are characterized as partially coherent [1]. Unlike passive radar (PR), this work does not assume the presence of a reference channel that measures the signal radiated by an illuminator of opportunity. Thus, in our terminology, this study focuses on *passive source localization* (PSL).

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The problem of PSL has been extensively studied in the literature; see [1] and references therein. The work in [1] considers Gaussian noise and deterministic signals, where the transmitted signal is treated as an unknown parameter to be estimated within the generalized likelihood ratio (GLR) framework, leading to a first-order model [8]. Specifically, it derives detectors based on the GLR for scenarios with known, unknown/equal, and unknown/unequal noise variances, as well as cases with coherent, partially coherent, and non-coherent channels. In contrast, [2] investigates PSL under the Gaussian assumption for both noise and signal. Thus, the distribution of the unknown signal can be marginalized out rather than estimated directly, resulting in a second-order model [8].

Unlike [2], which assumes spatially white transmitted signals, this work considers colored (or correlated) signals, thereby generalizing our previous results. However, this generalization results in a significantly more complex optimization problem for estimating the unknown parameters. In particular, there is no closed-form solution for the maximum likelihood estimates under the alternative hypothesis and it is necessary to resort to iterative approaches. In particular, we select an alternating optimization (AO) approach. The first step of the AO yields closed-form maximizers, whereas it is only possible to find closed-form solutions in the second step for a subset of parameters. The remaining ones are estimated via a minorization-maximization (MM) algorithm [9]. In particular, one closed-form step of the MM suffices to ensure the convergence of the AO, according to our experiments. Finally, Monte Carlo simulations are used to evaluate the performance of the proposed GLR with that of [1], [2], which yield similar conclusions to those in [10]. Basically, for highly colored signals, measured with the spectral flatness of the eigenvalues of the signal covariance matrix, the proposed GLR outperforms the competitors. This advantage degrades as the signal becomes increasingly white.

II. Signal Model

We consider an unknown bandlimited baseband signal $w(t)$, that is up-converted to a carrier frequency f_c. Assuming that, for the time being, the up-converted signal, $s(t) = \mathrm{Re}\{e^{j2\pi f_c} w(t)\}$, is transmitted through a free space channel, then a noisy, delayed, and scaled version is received at each sensor, which after down-conversion becomes

$$y_i(t) = G_i e^{-j2\pi f_c \tau_i(t)} w(t-\tau_i(t)) + r_i(t), \quad i=1,2.$$

Here, G_i is the channel gain between the emitter and the ith sensor, $r_i(t)$ is the sensor noise and $\tau_i(t)$ is a time-varying delay that encompasses the offset between the transmitter and receiver clocks, possible carrier frequency offsets, and the time-varying propagation delay from the emitter to the ith sensor. In this work we consider that the propagation delay is small, which allows us to approximate $\tau_i(t)$ as $\tau_i(t) \approx \tau_i + \nu_i t/f_c$, where $\nu_i t/f_c \ll \tau_i, \forall t$. Hence, we can approximate $y_i(t)$ as $y_i(t) = \tilde{\alpha}_i e^{j2\pi\nu_i t} w(t-\tau_i) + r_i(t)$, where $\tilde{\alpha}_i = G_i e^{-j2\pi f_c \tau_i}$. Synchronizing this signal in delay and Doppler, it becomes [1]

$$\begin{aligned} x_i(t) &= e^{-j2\pi\nu_i t} y_i(t+\tau_i) \\ &= \tilde{\alpha}_i e^{j2\pi\nu_i\tau_i} w(t) + e^{-j2\pi\nu_i t} r_i(t+\tau_i). \end{aligned}$$

Now, sampling $x_i(t)$ at a rate $1/T_s$ and stacking L measurements into the vector $\mathbf{x}_i$, we get

$$\mathbf{x}_i = \alpha_i \mathbf{w} + \mathbf{r}_i, \tag{1}$$

where $\alpha_i = \tilde{\alpha}_i e^{j2\pi\nu_i k_i T_s}$, $\mathbf{x}_i = [x_i[0], \dots, x_i[L-1]]^T, \mathbf{r}_i = [r_i[0], \dots, r_i[L-1]]^T$, and $\mathbf{w} = [w[0], \dots, w[L-1]]^T$, with $\tau_i = k_i T_s$ the delay, $x_i[n] = x_i(nT_s), w[n] = w(nT_s)$, and with some abuse of notation $r_i[n] = e^{-j2\pi\nu_i n T_s} r_i((n+k_i)T_s)$. Finally, we assume that L is such that $\nu_i T_s L/f_c \ll \tau_i$, the channel amplitude α_i is unknown and given no prior distribution, and the noise is assumed zero-mean Gaussian and white, $\mathbf{r}_i \sim \mathcal{CN}_L(\mathbf{0}, \sigma_i^2 \mathbf{I}_L)$, with unknown variance.

In this work, we consider subspace models for the transmitted signal, that is, $\mathbf{w} = \mathbf{U}\mathbf{s}$, where $\mathbf{U} \in \mathbb{C}^{L \times p}$ is an orthonormal basis for the known p-dimensional subspace $\langle \mathbf{U} \rangle$, and $\mathbf{s} \in \mathbb{C}^p$ are complex coefficients that determine the position of $\mathbf{w}$ in the subspace. Thus, the model (1) becomes $\mathbf{x}_i = \alpha_i \mathbf{U}\mathbf{s} + \mathbf{r}_i$, with $\mathbf{U}$ an arbitrary known basis, and α_i and $\mathbf{s}$ unknown. Considering the more general case of a linear passband channel, the received signal becomes [1]

$$\mathbf{x}_i = \alpha_i \mathbf{U}_i \mathbf{s} + \mathbf{r}_i, \tag{2}$$

and $\mathbf{U}_i$ accounts now for the subspace model for the signal and the linear channel.

Assuming that each sensor has access to N realizations of (2), the problem of deciding whether observations are only noise or a distorted version of the transmitted signal plus noise is defined as

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{x}_i[n] = \alpha_i \mathbf{U}_i \mathbf{s}[n] + \mathbf{r}_i[n], \\ \mathcal{H}_0 &: \mathbf{x}_i[n] = \mathbf{r}_i[n], \end{aligned} \tag{3}$$

with $i = 1, 2$, $n = 1, \dots, N$, $\mathbf{U}_i \in \mathbb{C}^{L \times p}$ represents the *known* basis for the ith subspace, $\alpha_i = g_i e^{j\phi_i}$ is the *unknown* complex amplitude for channel i, with $g_i \geq 0$, $\mathbf{s}[n] \in \mathbb{C}^p$ is the transmitted signal, which we model as $\mathbf{s}[n] \sim \mathcal{CN}_p(\mathbf{0}, \mathbf{Q})$, with $\mathbf{Q}$ an *unknown* positive definite covariance matrix without further structure.

III. Derivation of the GLRT

Taking into account that $\mathbf{s}[n] \sim \mathcal{CN}_p(\mathbf{0}, \mathbf{Q})$, the detection problem in (3) is a test for the covariance structure of the observations, i.e., a second-order model [8]. Concretely, under this Gaussianity assumption, (3) becomes

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{x}[n] \sim \mathcal{CN}_{2L}(\mathbf{0}, \mathbf{R}_1), \\ \mathcal{H}_0 &: \mathbf{x}[n] \sim \mathcal{CN}_{2L}(\mathbf{0}, \mathbf{R}_0), \end{aligned} \tag{4}$$

where $\mathbf{x}[n] = [\mathbf{x}_1^T[n] \ \mathbf{x}_2^T[n]]^T, n = 1, \dots, N$, the covariance matrix under $\mathcal{H}_0$ is

$$\mathbf{R}_0 = \begin{bmatrix} \sigma_1^2 \mathbf{I}_L & \mathbf{0} \\ \mathbf{0} & \sigma_2^2 \mathbf{I}_L \end{bmatrix},$$

and, assuming without loss of generality $\phi_1 = 0$, the covariance matrix under $\mathcal{H}_1$ is

$$\mathbf{R}_1 = \begin{bmatrix} g_1^2 \mathbf{U}_1 \mathbf{Q} \mathbf{U}_1^H + \sigma_1^2 \mathbf{I}_L & g_1 g_2 e^{-j\phi_2} \mathbf{U}_1 \mathbf{Q} \mathbf{U}_2^H \\ g_1 g_2 e^{j\phi_2} \mathbf{U}_2 \mathbf{Q} \mathbf{U}_1^H & g_2^2 \mathbf{U}_2 \mathbf{Q} \mathbf{U}_2^H + \sigma_2^2 \mathbf{I}_L \end{bmatrix}.$$

The GLR for the problem in (4) is

$$\Lambda = \frac{\max_{\mathbf{R}_1} \ell(\mathbf{R}_1; \mathbf{X})}{\max_{\mathbf{R}_0} \ell(\mathbf{R}_0; \mathbf{X})} = \frac{\ell(\hat{\mathbf{R}}_1; \mathbf{X})}{\ell(\hat{\mathbf{R}}_0; \mathbf{X})}, \tag{5}$$

where $\mathbf{X} = [\mathbf{X}_1^T \, \mathbf{X}_2^T]$, with $\mathbf{X}_i = [\mathbf{x}_i[1] \ \cdots \ \mathbf{x}_i[N]]$, and $\ell(\mathbf{R}_h; \mathbf{X})$ is the likelihood of the hth hypothesis, given by

$$\ell(\mathbf{R}_h; \mathbf{Y}) = \frac{1}{\pi^{2LN} \det(\mathbf{R}_h)^N} \exp\left\{-N \operatorname{tr}\left(\mathbf{R}_h^{-1} \mathbf{S}\right)\right\}.$$

Finally, the sample covariance matrix is

$$\mathbf{S} = \frac{1}{N} \sum_{n=1}^N \mathbf{x}[n] \mathbf{x}^H[n] = \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^H & \mathbf{S}_{22} \end{bmatrix},$$

and $\hat{\mathbf{R}}_h$ is the maximum likelihood (ML) estimate of the covariance matrix under hypothesis h.

A. ML estimates under $\mathcal{H}_0$

Under the null hypothesis, it is only necessary to estimate the noise variances σ_1^2 and σ_2^2, for which the ML estimates are $\hat{\sigma}_{i,0}^2 = \operatorname{tr}(\mathbf{S}_{ii})/L$. Therefore, the compressed log-likelihood becomes[1]

$$\log \ell(\hat{\mathbf{R}}_0; \mathbf{X}) = \log \ell(\hat{\sigma}_{1,0}^2, \hat{\sigma}_{2,0}^2) = -L \log \hat{\sigma}_{1,0}^2 - L \log \hat{\sigma}_{2,0}^2.$$

B. ML estimates under $\mathcal{H}_1$

To obtain the compressed likelihood, the first step is to rewrite the log-likelihood in a more amenable form. Concretely, the following lemma presents an expression for $\log \ell(\mathbf{R}_1; \mathbf{X})$ that is parametrized in alternative parameters.

Lemma 1: The log-likelihood can be written as

$$\begin{aligned} \log \ell(\mathbf{R}_1; \mathbf{X}) &= \log \ell(\beta_1, \beta_2, \phi_2, \rho_1, \rho_2, \mathbf{\Lambda}, \mathbf{V}) = \operatorname{tr}\left(\mathbf{\Psi} \mathbf{V}^H \mathbf{W} \mathbf{V}\right) \\ &+ \sum_{i=1}^2 \left[L \log \rho_i - \rho_i \operatorname{tr}(\mathbf{S}_{ii})\right] - \sum_{i=1}^p \log(1 + \lambda_i), \end{aligned} \tag{6}$$

[1]In all log-likelihoods in this paper, including the next one, constant and multiplicative terms that do not depend on data will be omitted.

where the precision variables are $\rho_i = 1/\sigma_i^2$, $\beta_i = g_i/\sigma_i$, $\mathbf{\Lambda}$ and $\mathbf{V}$ are the eigenvalues and eigenvectors matrices of $\mathbf{Q}$, respectively, that is, $\mathbf{Q} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, with $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \dots, \lambda_p)$ and $\lambda_i \geq \lambda_{i+1}$. Moreover, defining $\tilde{\mathbf{S}}_{ik} = \mathbf{U}_i^H \mathbf{S}_{ik} \mathbf{U}_k$, we can write

$$\mathbf{W} = \rho_1 \beta_1^2 \tilde{\mathbf{S}}_{11} + \rho_2 \beta_2^2 \tilde{\mathbf{S}}_{22} + \sqrt{\rho_1 \rho_2} \beta_1 \beta_2 \left(\tilde{\mathbf{S}}_{21} e^{-j\phi_2} + \tilde{\mathbf{S}}_{21}^H e^{j\phi_2} \right),$$

and

$$\mathbf{\Psi} = \mathrm{diag}\left(\frac{\lambda_1}{1+\lambda_1}, \dots, \frac{\lambda_p}{1+\lambda_p} \right).$$

Finally, due to the problem invariances we have assumed $\beta_1^2 + \beta_2^2 = 1$.

Proof: The determinant of $\mathbf{R}_1$ is $\det(\mathbf{R}_1) = \sigma_1^{2L}\sigma_2^{2L} \prod_{i=1}^p \left(1 + \lambda_i(\beta_1^2 + \beta_2^2)\right)$, and its inverse

$$\mathbf{R}_1^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2}\left(\mathbf{I}_L - \beta_1^2 \tilde{\mathbf{U}}_1 \mathbf{\Psi} \tilde{\mathbf{U}}_1^H \right) & -\frac{\beta_1\beta_2 e^{-j\phi_2}}{\sigma_1\sigma_2} \tilde{\mathbf{U}}_1 \mathbf{\Psi} \tilde{\mathbf{U}}_2^H \\ -\frac{\beta_1\beta_2 e^{j\phi_2}}{\sigma_1\sigma_2} \tilde{\mathbf{U}}_2 \mathbf{\Psi} \tilde{\mathbf{U}}_1^H & \frac{1}{\sigma_2^2}\left(\mathbf{I}_L - \beta_2^2 \tilde{\mathbf{U}}_2 \mathbf{\Psi} \tilde{\mathbf{U}}_2^H \right) \end{bmatrix},$$

where $\tilde{\mathbf{U}}_i = \mathbf{U}_i \mathbf{V}$. The proof follows by straightforward manipulations. □

Lemma 1 provides the log-likelihood under $\mathcal{H}_1$ as a function of $\{\beta_1, \beta_2, \phi_2, \rho_1, \rho_2, \mathbf{\Lambda}, \mathbf{V}\}$, which is an invertible transformation of the original parameters $\{g_1, g_2, \phi_2, \sigma_1^2, \sigma_2^2, \mathbf{\Lambda}, \mathbf{V}\}$.

a) Step 1 - Optimization in $\mathbf{\Lambda}$ and $\mathbf{V}$: In the first step, we maximize (6) with respect to $\mathbf{\Lambda}$ and $\mathbf{V}$, considering the remaining parameters fixed. The ML estimates of $\mathbf{\Lambda}$ and $\mathbf{V}$ are presented in the next lemma.

Lemma 2: Denoting the eigendecomposition of $\mathbf{W}$ as $\mathbf{W} = \mathbf{E}\mathbf{\Gamma}\mathbf{E}^H$, with $\mathbf{\Gamma} = \mathrm{diag}(\gamma_1, \dots, \gamma_p)$ and $\gamma_i \geq \gamma_{i+1}$, the ML estimates are $\hat{\mathbf{V}} = \mathbf{E}$ and $\hat{\lambda}_i = \max(\gamma_i - 1, 0)$. Therefore, $\hat{\mathbf{Q}} = \mathbf{E}\left[\mathbf{\Gamma} - \mathbf{I}_p\right]_+ \mathbf{E}^H$, where $[\cdot] = \max(\cdot, 0)$.

Proof: From [8, Sec. B.9.1], it follows that $\hat{\mathbf{V}} = \mathbf{E}$. Plugging this back into (6), the compressed log-likelihood becomes

$$\log \ell(\beta_1, \beta_2, \phi_2, \rho_1, \rho_2, \mathbf{\Lambda}, \hat{\mathbf{V}}) = \sum_{i=1}^p \frac{\lambda_i \gamma_i}{1+\lambda_i} + \sum_{i=1}^2 \left[L \log \rho_i - \rho_i \,\mathrm{tr}\left(\mathbf{S}_{ii}\right) \right] - \sum_{i=1}^p \log(1+\lambda_i). \quad (7)$$

Finally, by Lagrangian optimization, it is easy to find $\hat{\lambda}_i$, as they must be non-negative. □

b) Step 2 - Optimization in $\beta_1, \beta_2, \phi_2, \rho_1$, and ρ_2: The second step in the alternating optimization seeks the ML estimates of $\beta_1, \beta_2, \phi_2, \rho_1$, and ρ_2, for fixed $\mathbf{\Lambda}$ and $\mathbf{V}$. In particular, for β_1, β_2, and ϕ_2, it is possible to find closed-form estimates, but for ρ_1 and ρ_2, we will resort to a majorization-minimization (MM) approach [9]. Lemma 3 obtains the estimates of β_1, β_2, and ϕ_2, whereas the update rules for the MM are presented in Lemma 4.

Lemma 3: The ML estimates of ϕ_2 and $\beta_i, i = 1, 2$, are $\hat{\phi}_2 = \psi_{21}$ and $[\hat{\beta}_1 \;\; \hat{\beta}_2]^T$ is the principal eigenvector of

$$\mathbf{\Sigma} = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^{1/2} \begin{bmatrix} \eta_{11} & \eta_{21} \\ \eta_{21} & \eta_{22} \end{bmatrix} \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix}^{1/2}, \quad (8)$$

normalized such that $\hat{\beta}_1^2 + \hat{\beta}_2^2 = 1$.

Proof: The proof starts by rewriting (6) as

$$\log \ell(\beta_1, \beta_2, \phi_2, \rho_1, \rho_2, \mathbf{\Lambda}, \mathbf{V}) = -\sum_{i=1}^p \log(1+\lambda_i) + \sum_{i=1}^2 \left[L \log \rho_i - \rho_i \,\mathrm{tr}\left(\mathbf{S}_{ii}\right) + \rho_i \beta_i^2 \eta_{ii} \right] + 2\sqrt{\rho_1\rho_2}\beta_1\beta_2\eta_{21} \cos\left(\psi_{21} - \phi_2\right), \quad (9)$$

where we have defined $\eta_{ii} = \mathrm{tr}\left(\mathbf{\Psi}\mathbf{V}^H \tilde{\mathbf{S}}_{ii} \mathbf{V}\right)$ and $\eta_{21} e^{j\psi_{21}} = \mathrm{tr}\left(\mathbf{\Psi}\mathbf{V}^H \tilde{\mathbf{S}}_{21} \mathbf{V}\right)$. It is easy to verify that the maximizer of (9) with respect to ϕ_2 is $\hat{\phi}_2 = \psi_{21}$, which yields

$$\log \ell(\beta_1, \beta_2, \hat{\phi}_2, \rho_1, \rho_2, \mathbf{\Lambda}, \mathbf{V}) = -\sum_{i=1}^p \log(1+\lambda_i) + \sum_{i=1}^2 \left[L \log \rho_i - \rho_i \,\mathrm{tr}\left(\mathbf{S}_{ii}\right) \right] + \boldsymbol{\beta}^T \mathbf{\Sigma} \boldsymbol{\beta}, \quad (10)$$

where $\boldsymbol{\beta} = [\beta_1 \;\; \beta_2]^T$. The maximizer of (10) is the principal normalized eigenvector of $\mathbf{\Sigma}$, and the compressed log-likelihood becomes

$$\log \ell(\hat{\beta}_1, \hat{\beta}_2, \hat{\phi}_2, \rho_1, \rho_2, \mathbf{\Lambda}, \mathbf{V}) = -\sum_{i=1}^p \log(1+\lambda_i) + \sum_{i=1}^2 \left[L \log \rho_i - \rho_i \,\mathrm{tr}\left(\mathbf{S}_{ii}\right) \right] + \sigma_{max}. \quad (11)$$

□

Lemma 4: In each step of the MM algorithm, we update the precision variables as

$$\hat{\rho}_i^{(k+1)} = \max\left(\frac{L}{\mathrm{tr}\left(\mathbf{S}_{ii}\right) - \Gamma_i^{(k)}}, 0 \right),$$

where

$$\Gamma_i^{(k)} = \left. \frac{\partial \sigma_{max}}{\partial \rho_i} \right|_{\substack{\rho_1 = \rho_1^{(k)} \\ \rho_2 = \rho_2^{(k)}}},$$

with σ_{max} being the principal eigenvalue of $\mathbf{\Sigma}$.

Proof: It is easy to show that σ_{max} in (11) is convex in ρ_1 and ρ_2. Then, (11) can be minorized by a first-order Taylor series, that is,

$$\log \ell(\hat{\beta}_1, \hat{\beta}_2, \hat{\phi}_2, \rho_1, \rho_2, \mathbf{\Lambda}, \mathbf{V}) \geq -\sum_{i=1}^p \log(1+\lambda_i) + \sigma_{max}^{(k)} + \sum_{i=1}^2 \left[L \log \rho_i - \rho_i \,\mathrm{tr}\left(\mathbf{S}_{ii}\right) + \Gamma_i^{(k)} \left(\rho_i - \rho_i^{(k)} \right) \right], \quad (12)$$

where $\sigma_{max}^{(k)}$ is the principal eigenvalue of $\mathbf{\Sigma}$ evaluated at $\rho_1^{(k)}$ and $\rho_2^{(k)}$. The proof concludes by Lagrangian optimization of (12). □

Finally, the compressed log-likelihood $\log \ell(\hat{\mathbf{R}}_1; \mathbf{X}) = \log \ell(\hat{\beta}_1, \hat{\beta}_2, \hat{\phi}_2, \hat{\rho}_1, \hat{\rho}_2, \hat{\mathbf{\Lambda}}, \hat{\mathbf{V}})$ is obtained by plugging the values of the unknown parameters after the convergence of the alternating optimization approach in either (6) or in (9).

C. GLR and practical details

Given the compressed log-likelihoods obtained in the two previous subsections, the log-GLR is given by

$$\log \Lambda = \log \ell(\hat{\mathbf{R}}_1; \mathbf{X}) - \log \ell(\hat{\mathbf{R}}_0; \mathbf{X}).$$

Moreover, the iterative nature of the estimators under $\mathcal{H}_1$ necessitates an initialization, which will be selected in the experiments as $\mathbf{Q} = \mathbf{0}$, $\phi_2 = 0$, $\beta_1 = \beta_2 = 0.5$, and

$$\rho_i^{(0)} = \left[\frac{1}{L-p} \operatorname{tr}\left(\mathbf{P}_{\mathbf{U}_i}^{\perp} \mathbf{S}_{ii} \right) \right]^{-1},$$

where $\mathbf{P}_{\mathbf{U}_i}^{\perp} = \mathbf{I}_L - \mathbf{U}_i \mathbf{U}_i^H$ is the projector onto the orthogonal subspace to $\langle \mathbf{U}_i \rangle$. Finally, in the numerical results, one iteration of the MM procedure suffices to ensure the convergence of the alternating optimization.

IV. Numerical results

In this section, using Monte Carlo simulations, we evaluate the performance of the GLR, Λ, and compare it with our previous work in [2], which is the GLR for white Gaussian signals, $\mathbf{s}[n] \sim \mathcal{CN}_p(\mathbf{0}, \mathbf{I}_p)$. We denote this GLR by Λ_w. An additional competitor given by the first-order counterpart, $\mathcal{L}$, is included in the comparison. Concretely, it does not assume any prior distribution for $\mathbf{s}[n]$ and it is therefore estimated using the ML framework [1]. However, since there are also no closed-form solution for the ML estimates of the noise variances, [1] proposes to use $1/\rho_i^{(0)}$.

In the following experiments, we consider a setup with $N = 20$, $L = 6$, and $p = 4$. In each Monte Carlo simulation, the subspace bases $\mathbf{U}_i$ are generated uniformly on the Stiefel manifold, the channel gains are $\alpha_i \sim \mathcal{CN}_1(0, 1)$, and the signal covariance matrix is also randomly generated as $\mathbf{Q} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^H$, where $\mathbf{V}$ is drawn uniformly from the unitary group and $\lambda_i = a^{i-1}, i = 1, \ldots, p$. This approach allows us to control the "degree of color" of the transmitted signal. In particular, we choose the spectral flatness as the degree measure, which is given by

$$\eta = \frac{\left(\prod_{i=1}^{p} \lambda_i\right)^{1/p}}{\frac{1}{p}\sum_{i=1}^{p} \lambda_i}.$$

Admittedly, the spectral flatness and the exponential profile only partially capture the color of the source, but we believe it is an insightful alternative. When $\eta = 1$, the signal is white and the correlation increases as η decreases. Finally, the noise variances are selected to achieve the desired signal-to-noise ratio (SNR) of the corresponding channel, which is defined as

$$\text{SNR}_i = 10 \log_{10} \left(\frac{|\alpha_i|^2 \operatorname{tr}(\mathbf{Q})}{\sigma_i^2 L} \right).$$

In the first experiment, we obtain the probability of missed detection, p_m, for a fixed probability of false alarm $p_{fa} = 10^{-3}$ of the three aforementioned detectors for $\eta = 0.2$, varying SNR of channel 1 and two values for the SNR of the second channel: $\text{SNR}_2 = \text{SNR}_1$ and $\text{SNR}_2 = \text{SNR}_1 + 10$ dBs. The results for this experiment are shown in Fig. 1, where we can see that the proposed GLR, Λ, outperforms both competitors, the second-order GLR for white signals, Λ_w, and the first-order GLR, $\mathcal{L}$. Interestingly, the performance of Λ_w is not severely degraded for this highly colored scenario, and it even outperforms the first-order GLR, which makes no assumption about the signal covariance matrix. Moreover, it seems that for large differences in the SNRs of both channels, there is a large performance improvement of Λ over $\mathcal{L}$ and Λ_w, while for smaller differences the gap between $\mathcal{L}$ and Λ_w increases.

The second experiment considers the same scenario, with the exception of the spectral flatness, which is now $\eta = 0.5$. Fig. 2 depicts again the probability of missed detection vs. SNR_1 for $p_{fa} = 10^{-3}$ and the same values of SNR_2. Similar conclusions can be drawn from this experiment, but now the differences between the detectors are reduced due to the smaller degree of color (larger η).

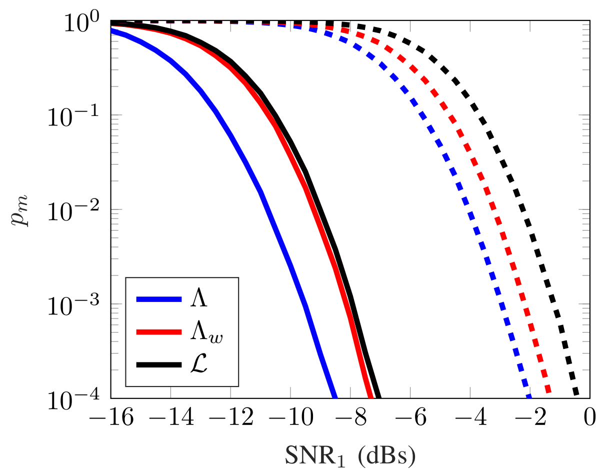

Fig. 1. Probability of missed detection vs. SNR_1 for $p_{fa} = 10^{-3}$ in a scenario with $N = 20, L = 6, p = 4, \eta = 0.2$, and two different cases for SNR_2. 1) Dashed line: $\text{SNR}_2 = \text{SNR}_1$; 2) Solid line: $\text{SNR}_2 = \text{SNR}_1 + 10$

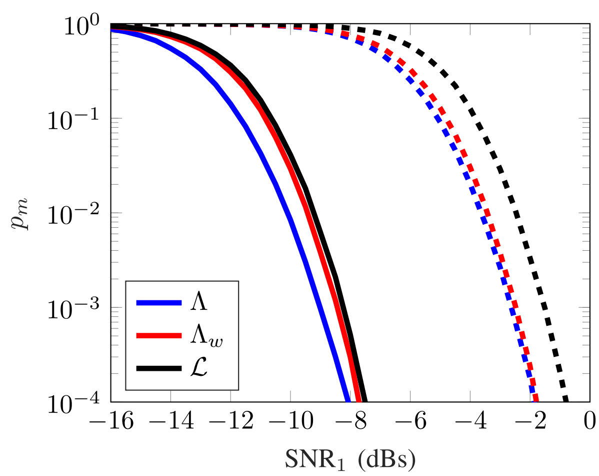

Fig. 2. Probability of missed detection vs. SNR_1 for $p_{fa} = 10^{-3}$ in a scenario with $N = 20, L = 6, p = 4, \eta = 0.5$, and two different cases for SNR_2. 1) Dashed line: $\text{SNR}_2 = \text{SNR}_1$; 2) Solid line: $\text{SNR}_2 = \text{SNR}_1 + 10$

V. Conclusions

This study addressed signal detection in passive sensor arrays by deriving the generalized likelihood ratio (GLR) framework for detecting colored Gaussian subspace signals transmitted through two unknown channels when the signal subspaces are known. The signals were received at two sensors and contaminated by additive white Gaussian noise of unknown variance. By formulating the detection problem for unknown signal covariance, we introduced an alternating optimization algorithm—augmented with a single minorization-maximization (MM) iteration—to compute maximum likelihood estimates under the alternative hypothesis. This approach was necessary by the analytical intractability of the problem, which precluded closed-form solutions.

The proposed second-order GLR's performance was rigorously evaluated through numerical experiments and compared against two reference detectors: 1) a second-order GLR designed for white Gaussian signals [2] and 2) a first-order GLR that imposes no distributional assumptions on the transmitted signal [1]. Empirical results demonstrated that the proposed detector achieves superior performance for highly correlated signals, as quantified by the spectral flatness of the eigenvalues of the signal covariance matrix. However, as correlation decreases, the white-signal GLR exhibits enhanced detection capability—a phenomenon attributed to the over-parametrization in scenarios with increased signal structure.

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