A GENERAL TEST FOR THE LINEAR STRUCTURE OF COVARIANCE MATRICES OF GAUSSIAN POPULATIONS

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ABSTRACT

This paper addresses the problem of testing whether a covariance matrix can be expressed by an unknown linear combination of a set of known matrices or by another unknown linear combination of a set of different, but known, matrices. This problem is of interest in a wide range of real-world applications, such as radar, sonar, and spectrum sensing. We study the problem under the Gaussian assumption and derive the generalized likelihood ratio test (GLRT). Since there is no general closed-form solution for the maximum likelihood (ML) estimates of the covariance matrices, which are required for the GLRT, we resort to a powerful inverse iteration algorithm. Finally, an example, along with numerical results, is given to illustrate the methodology.

Index Terms— Covariance structure, generalized likelihood ratio test (GLRT), hypothesis testing, inverse iteration algorithm.

1. INTRODUCTION

Detecting a signal contaminated by noise is a problem that plays an important role in a broad range of applications, such as target detection in radar [1] and sonar [2], image processing [3], and spectrum sensing [4, 5], to name a few. The main property that allows the aforementioned detection problem to be solved is the difference between the spatio-temporal structure of the signal of interest and that of the noise, which results in received signals with different covariance structures.

The structures that typically appear in detection problems are quite diverse. For instance, the problem in [6] considered a uniform linear array (ULA), which translates into signals with a stationary spatial correlation and, therefore, the covariance matrix is Toeplitz. However, if the array consists of different well-separated sub-arrays, the covariance matrix becomes block-diagonal, as the correlation between sub-arrays can be considered negligible [7]. Similarly, for wide-sense stationary (WSS) univariate time series, the covariance matrix of the stack of n observations is also Toeplitz and it is block-Toeplitz for multivariate WSS signals [4, 8].

Interestingly, all aforementioned structures, and many more, can be represented by an *unknown* linear combination of *known* matrices (the basis). Thus, we face a detection problem with unknown parameters, i.e., composite hypotheses [9]. To address hypothesis tests with composite hypotheses, the common approach is based on the generalized likelihood ratio test (GLRT). The derivation of the GLRT requires the maximum likelihood (ML) estimates of the covariance matrices, which may not have a closed-form solution, even for Gaussian signals. This has motivated the development of ad-hoc tests, such as those in [10] for spectrum sensing problems.

In this paper, we develop a testing procedure for the aforementioned general linear class of covariance structures of multivariate Gaussian signals based on the GLRT. Since there is no closed-form solution for the ML estimates of the unknown parameters, we resort to the inverse iteration algorithm (IIA) [11], which is a powerful algorithm for computing the ML estimates of structured covariance matrices. As an interesting example of the proposed detector, we study the problem of testing whether the covariance matrix of the observations is block-Toeplitz. A similar problem was studied in [8, 12], where the authors exploited the asymptotic equivalence of block-Toeplitz matrices with block-circulant ones, for which there exist closed-form ML estimates. However, such an approach is only applicable for testing block-Toeplitz structure with different block sizes. Thus, it is not easy to generalize it to detect block-Toeplitz structure against other linear structures, as the preprocessing steps might alter the other structure. Moreover, in the finite case, the equivalence between block-Toeplitz and block-circulant matrices might be poor, which would result in noisy ML estimates. Here, by resorting to the inverse iteration method, we derive the exact (i.e., non-asymptotic) GLRT, thereby allowing us to test, for instance, between block-Toeplitz and more general linear structures in the finite case.

We also study the distributions of the proposed GLRT under both hypotheses, showing that, when $\mathcal{H}_1$ includes $\mathcal{H}_0$, they are asymptotically central and non-central chi-squared distributions under the null and the alternative, respectively. Finally, we evaluate the performance of the proposed detector by means of Monte Carlo simulations. First, we show that it presents the same performance for scenarios where there are closed-form GLRT statistics. Second, for a particular detection problem, we demonstrate that it performs better than the competitors. Third, we provide numerical results to illustrate the accuracy of the proposed distributions.

2. DETECTION PROBLEM

Suppose $\mathbf{x} \in \mathbb{C}^p$ follows a zero-mean circular complex Gaussian distribution $\mathcal{CN}(\mathbf{0}, \mathbf{\Sigma})$. Under the null hypothesis ($\mathcal{H}_0$), $\mathbf{\Sigma}$ can be represented by a linear combination of the basis $\{\mathbf{B}_1, \ldots, \mathbf{B}_r\}$, that

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is,

$$\mathbf{\Sigma}_0 = \sum_{i=1}^{r} \alpha_i \mathbf{B}_i, \tag{1}$$

where α_i are the *unknown* coefficients of the linear combination and $\mathbf{B}_i \in \mathbb{R}^{p\times p}$ are *known* matrices. Similarly, under the alternative hypotheses ($\mathcal{H}_1$), the covariance matrix can be represented by another basis $\{\mathbf{D}_1, \ldots, \mathbf{D}_s\}$ as

$$\mathbf{\Sigma}_1 = \sum_{i=1}^{s} \beta_i \mathbf{D}_i,$$

where β_i are also unknown coefficients and $\mathbf{D}_i \in \mathbb{R}^{p\times p}$ are known matrices.

Collecting now n samples of $\mathbf{x}$ in the data matrix $\mathbf{X} = [\mathbf{x}(1), \ldots, \mathbf{x}(n)]$, our aim is to test between the following hypotheses:

$$\mathcal{H}_0 : \mathbf{\Sigma} \in \mathbb{B},$$
$$\mathcal{H}_1 : \mathbf{\Sigma} \in \mathbb{D},$$

where $\mathbb{B}$ is the set of matrices in the span of $\{\mathbf{B}_1, \ldots, \mathbf{B}_r\}$ and $\mathbb{D}$ that of $\{\mathbf{D}_1, \ldots, \mathbf{D}_s\}$. Without loss of generality, we assume that $r < s$. That is, $\mathcal{H}_0$ is more structured than $\mathcal{H}_1$.

In the subsequent sections, we derive the generalized likelihood ratio test (GLRT) for this detection problem, and establish its limiting distributions, which are necessary for threshold selection and performance evaluation.

3. THE GENERALIZED LIKELIHOOD RATIO TEST

3.1. Derivation of GLRT

Under the Gaussian assumption, the probability density function (PDF) of $\mathbf{x}$ under both hypotheses is given by

$$f(\mathbf{X}|\mathbf{\Sigma}_i) = \frac{1}{\pi^{pn}\det^n(\mathbf{\Sigma}_i)} \exp\left[-n\text{tr}\left(\mathbf{\Sigma}_i^{-1}\mathbf{S}\right)\right],$$

where $\text{tr}(\cdot)$ and $\det(\cdot)$ represent the trace and determinant operators, and $\mathbf{S} = \mathbf{X}\mathbf{X}^H/n$ is the sample covariance matrix (SCM). According to the GLR criterion [9], the unknown quantities in the likelihood ratio, namely the covariance matrices, are replaced by their ML estimates under the corresponding hypothesis. Thus, the GLRT can be written as

$$\begin{aligned} T_{\text{GLRT}} &= \frac{\sup_{\mathbf{\Sigma}_1\in\mathbb{D}} f(\mathbf{X}|\mathbf{\Sigma}_1)}{\sup_{\mathbf{\Sigma}_0\in\mathbb{B}} f(\mathbf{X}|\mathbf{\Sigma}_0)} \\ &= \frac{\det^n(\hat{\mathbf{\Sigma}}_0)}{\det^n(\hat{\mathbf{\Sigma}}_1)} \exp\left\{-n\text{tr}\left[\left(\hat{\mathbf{\Sigma}}_1^{-1} - \hat{\mathbf{\Sigma}}_0^{-1}\right)\mathbf{S}\right]\right\}, \end{aligned} \tag{2}$$

where $\hat{\mathbf{\Sigma}}_1$ and $\hat{\mathbf{\Sigma}}_0$ are, respectively, the ML estimates of $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_0$.

Before computing the ML estimates, let us simplify the GLRT in (2). It is easy to see from (1), that for any matrix $\mathbf{\Sigma}$ in $\mathbb{B}$ (or $\mathbb{D}$), $a\mathbf{\Sigma}$ also belongs to $\mathbb{B}$ (or $\mathbb{D}$), i.e., $\mathbb{B}$ and $\mathbb{D}$ are cones. Therefore, we may use a lemma from [13], which sets a trace constraint on the ML estimate of covariance matrices that belong to a cone. Concretely, this lemma states the following.

Lemma 1 *If a covariance matrix $\mathbf{\Sigma}$ belongs to a cone, its ML estimate within such cone satisfies*

$$\text{tr}\left(\hat{\mathbf{\Sigma}}^{-1}\mathbf{S}\right) = p,$$

where p is the dimension of the random vector.

Algorithm 1 Inverse iteration algorithm

Require: $\{\mathbf{B}_0, \ldots, \mathbf{B}_r\}$ and $\mathbf{S}$

Set $k = 0$ and $\mathbf{G}_{(0)} = \mathbf{I}_p$
repeat
  Compute $\mathbf{A} = \{[\mathbf{A}]_{i,j}\}_{i,j=1}^r$ and $\mathbf{c} = [c_1, \ldots, c_r]^T$ as:
    $[\mathbf{A}]_{i,j} = \text{tr}\left(\mathbf{G}_{(k)}^{-1}\mathbf{B}_i\mathbf{G}_{(k)}^{-1}\mathbf{B}_j\right)$
    $c_i = \text{tr}\left(\mathbf{G}_{(k)}^{-1}\mathbf{S}\mathbf{G}_{(k)}^{-1}\mathbf{B}_i\right)$
  Obtain $\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{c}$
  Calculate $\mathbf{G}_{(k+1)} = \sum_{i=1}^r \alpha_i \mathbf{B}_i$
  while $\mathbf{G}_{(k+1)} \not\succ 0$ **do**
    $\mathbf{G}_{(k+1)} = \left(\mathbf{G}_{(k+1)} + \mathbf{G}_{(k)}\right)/2$
  end while
  Update $k = k + 1$
until convergence
return $\hat{\mathbf{\Sigma}}_0 = \mathbf{G}_{(k)}$

Applying Lemma 1, the GLRT can be simplified as

$$T_{\text{GLRT}}^{1/n} = \frac{\det(\hat{\mathbf{\Sigma}}_0)}{\det(\hat{\mathbf{\Sigma}}_1)}. \tag{3}$$

Now, the last ingredient necessary for the GLRT is to compute the ML estimates of the covariance matrices. In general, there is no closed-form ML estimate for the covariance matrices $\mathbf{\Sigma}_0 \in \mathbb{B}$ and $\mathbf{\Sigma}_1 \in \mathbb{D}$. Some exceptions would be, for instance, when $\mathbf{\Sigma}_i$ is a positive definite matrix without further structure or it is a diagonal matrix. Thus, in this work, we resort to the inverse iteration algorithm [11]. This method is presented in Algorithm 1 for the estimation of $\mathbf{\Sigma}_0$. As in [14], since $\hat{\mathbf{\Sigma}}$ affects the value of the detector only through its determinant, we may set the convergence condition as

$$\left|\frac{\det(\mathbf{G}_{(k+1)})}{\det(\mathbf{G}_{(k)})} - 1\right| \leq \epsilon$$

with ϵ being an acceptable error rate. Finally, the ML estimates $\hat{\mathbf{\Sigma}}_0$ and $\hat{\mathbf{\Sigma}}_1$ obtained using the inverse iteration algorithm are plugged back in (3) to compute the GLRT.

3.2. Asymptotic distributions of the GLRT

Deriving the distributions of the GLRT is, in general, cumbersome, even in the asymptotic case. However, if $\mathcal{H}_1$ includes $\mathcal{H}_0$ as a particular case, we may obtain the asymptotic distribution of the GLRT using Wilks' theorem [15]. This is typically the case of most real-world applications. Concretely, Wilks's theorem states the following:

Theorem 1 (Wilks) *Consider the binary hypothesis testing problem*

$$\mathcal{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s,$$
$$\mathcal{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_{r_0}, \boldsymbol{\theta}_s,$$

where $\boldsymbol{\theta}_s \in \mathbb{R}^{g\times 1}$ is the nuisance parameter vector and $\boldsymbol{\theta}_{r_0} \in \mathbb{R}^{f\times 1}$. Then, the GLRT is asymptotically distributed as

$$2\ln(T_{\text{GLRT}}) \sim \begin{cases} \chi_f^2 & \text{under } \mathcal{H}_0 \\ \chi_f^2(\sigma^2) & \text{under } \mathcal{H}_1, \end{cases} \tag{4}$$

where χ_f^2 and $\chi_f^2(\sigma^2)$ denote central and non-central chi-squared PDFs, respectively, with f being the degrees of freedom and σ^2 the noncentrality parameter, which depends on the Fisher information matrix and the true parameters.

Although Theorem 1 provides a distribution for both hypotheses, in most problems it is not possible to obtain the noncentrality parameter since the computation of the Fisher information matrix is usually intractable. Thus, we may use the approximation of [14], which allows us to obtain the noncentrality parameter as

$$\sigma^2 = 2\ln(T_{\mathrm{GLRT}})|_{\hat{\boldsymbol{\theta}}=\boldsymbol{\theta}_1}, \tag{5}$$

where $\boldsymbol{\theta}_1$ are the parameters under $\mathcal{H}_1$. Then, using Theorem 1 and (5), it is easy to conclude that, in our problem, the degrees of freedom are $f = s - r$ under both hypotheses and σ^2 becomes

$$\begin{aligned}\sigma^2 &= 2\ln(T_{\mathrm{GLRT}})|_{\mathbf{S}=\boldsymbol{\Sigma}_1} \\ &= 2n\ln(\det(\bar{\boldsymbol{\Sigma}}_1)) - 2n\ln(\det(\boldsymbol{\Sigma}_1)),\end{aligned}$$

where $\bar{\boldsymbol{\Sigma}}_1$ is the output of Algorithm 1 when we use the basis $\mathbf{B}_1, \ldots, \mathbf{B}_s$ and $\boldsymbol{\Sigma}_1$ instead of $\mathbf{S}$.

3.3. Computational Complexity

The computational complexity of the iterative-GLRT mainly consists of three parts: the computation of the SCM, the determinants in (3) and the inverse iteration algorithm. It is well-known that the first two require $\mathcal{O}(p^2 n)$ and $\mathcal{O}(p^3)$ flops, respectively. For the inverse iteration algorithm, one of the most computationally demanding steps is the computation of the elements of $\mathbf{A}$ and $\mathbf{c}$, which requires $\mathcal{O}(p^3)$ flops per element, with a total of $r(r+1)/2+r$ elements. In addition, the algorithm also computes a matrix inverse in Step 4, which demands $\mathcal{O}(p^3)$ flops. Besides, Step 4 generates another $\mathcal{O}(p^3)$ flops. Thus, neglecting the computational complexity in other steps of the algorithm, the total complexity per iteration is

$$t(r) = \mathcal{O}\left(\left[\frac{r(r+1)}{2} + r\right] p^3 + 2p^3\right).$$

Consequently, the total amount of flops of the GLRT is:

$$t_{\mathrm{sum}} = l_0 \cdot t(r) + l_1 \cdot t(s) + \mathcal{O}\left(p^2 n + p^3\right), \tag{6}$$

where l_0 and l_1 are the number of iterations of Algorithm 1 under $\mathcal{H}_0$ and $\mathcal{H}_1$, which, according to our simulations, barely exceed 6 iterations. Obviously, the computational load of the inverse iteration algorithm grows rapidly with the sizes of the covariance matrix and the basis. Therefore, the simplification of the inverse iteration algorithm is still an interesting research line for future work.

4. APPLICATION OF THE PROPOSED GLRT

In this section, we present an interesting example to illustrate the applicability of the considered detection problem and the proposed iterative-GLRT. In particular, we consider that one covariance matrix is block-spherical, that is, $\boldsymbol{\Sigma}_0 = \mathbf{I} \otimes \boldsymbol{\Delta}$, with $\otimes$ denoting the Kronecker product, whereas the other one is block-Toeplitz with block size k. This problem appears, for instance, in the detection problem of multivariate temporally colored signals contaminated by temporally white noises with spatial (unknown) covariance matrix $\boldsymbol{\Delta}$.

Since closed-form ML estimates do not exist for block-Toeplitz matrices, [8] exploits the asymptotic equivalence of the block-Toeplitz matrices and block-circulant matrices, for which there exists an ML estimate based on the Fourier transform. However, in the non-asymptotic regime, we do not know how this estimate will perform. Nevertheless, since block-Toeplitz and block-spherical matrices can be represented by a linear combination of two sets of matrices, we may apply the proposed iterative-GLRT to solve this detection problem.

For a p-dimensional block-spherical matrix with block size k, the basis matrices $\mathbf{B}_1, \ldots, \mathbf{B}_{k^2}$ are block-spherical with the first diagonal block given by

$$[\mathbf{B}_l]_{i,j} = \delta(i,x)\delta(j,y) + \delta(i,y)\delta(j,x),$$

for $1 \leq l \leq k(k+1)/2$, and

$$[\mathbf{B}_l]_{i,j} = \imath\delta(i,x')\delta(j,y') - \imath\delta(i,y')\delta(j,x'),$$

for $k(k+1)/2 < l \leq k^2$. Here, $\imath = \sqrt{-1}$, $\delta(\cdot,\cdot)$ is the Kronecker delta function and $1 \leq i \leq k, 1 \leq j \leq k$. Moreover, x and y are two integers, with $1 \leq y \leq x \leq k$, such that $x(x-1)/2 + y = l$. Similarly, x' and y' are two integers, with $1 < y' < x' < k$, such that $(x'-1)(x'-2)/2 + y' = l - k(k+1)/2$.

Similarly, for block-Toeplitz matrices with block size k, the basis is defined by the block-Toeplitz matrices $\mathbf{D}_1, \ldots, \mathbf{D}_{2kp-k^2}$, where $\mathbf{D}_l = \mathbf{B}_l$ for $l = 1, \cdots, k^2$ and $\mathbf{D}_{k^2+1}, \cdots, \mathbf{D}_{2kp-k^2}$ have their first k columns as:

$$\begin{aligned}
[\mathbf{D}_l]_{i,j} &= \delta(i-k, l-s_0)\delta(j,1), && s_0 < l \leq s_1, \\
&\vdots \\
[\mathbf{D}_l]_{i,j} &= \delta\left(i-k, l-s_{k-1}\right)\delta(j,k), && s_{k-1} < l \leq s_k, \\
[\mathbf{D}_l]_{i,j} &= \imath\delta(i-k, l-r_0)\delta(j,1), && r_0 < l \leq r_1, \\
&\vdots \\
[\mathbf{D}_l]_{i,j} &= \imath\delta\left(i-k, l-r_{k-1}\right)\delta(j,k), && r_{k-1} < l \leq r_k,
\end{aligned}$$

where $q = p/k$, $s_m = k^2 + m(p-k)$, $r_m = s_k + m(p-k)$ and $1 \leq i \leq p, 1 \leq j \leq k$.

Finally, once these basis are defined, it is straightforward to compute the iterative-GLRT.

5. SIMULATION

In this section, we carry out Monte Carlo simulations to validate our theoretical findings. We first verify the correctness of the ML estimates obtained by Algorithm 1 by comparing the iterative-GLRT statistic with the statistic of several well-known GLRTs for detection problems where there are closed-form solutions. Then, we consider the problem of testing block-Toeplitz against block-spherical covariance structures, which is the problem introduced in the previous section. Finally, we check the accuracy of the asymptotic distributions in (4) under $\mathcal{H}_0$ and $\mathcal{H}_1$.

5.1. Approximation error

In this section we collect the largest relative error of the iterative-GLRT (with a maximum of 10 iterations) with respect to several well-known closed-form GLRTs. Concretely, we ran 10^4 trials for the following GLRTs:

- The sphericity test (ST) proposed in [16].
- The Hadamard ratio test (HDM) proposed in [17].
- The sphericity test (BS) and the independence test (IN) for Gaussian vectors proposed in [18].

Table 1 lists the largest relative error of the iterative-GLRT in approximating the values of the exact GLRTs, where p denotes the size of the observations of the sphericity and Hadamard ratio tests, q denotes the number of vectors of size m of the tests for Gaussian vectors, and n is the number of observations. It can be observed that the errors are negligible for all four cases, which implies that the proposed iterative-GLRT converges to the exact value of the GLRT.

Detector	Parameters	Error
ST	$[p, n] = [9, 40]$	8.88×10^{-16}
HDM	$[p, n] = [9, 40]$	2.22×10^{-16}
BS	$[m, q, n] = [3, 3, 40]$	1.11×10^{-15}
IN	$[m, q, n] = [3, 3, 40]$	8.88×10^{-16}

Table 1. Largest relative error in 10^4 trials

5.2. Detection performance

Now we compare the performance of the iterative-GLRT with those of the asymptotic-GLRT and LMPIT-inspired test of [12] in a multichannel detection problem of stationary temporally colored signals contaminated by temporally white noise. More specifically, we consider the following model

$$\mathbf{y}(t) = \sum_{j=1}^{d} \mathbf{h}_j s_j(t) + \mathbf{v}(t),$$

where $\mathbf{y}(t) \in \mathbb{C}^k$ are the observations, $\mathbf{v}(t) \in \mathbb{C}^k$ is a temporally white noise with covariance matrix $\mathbf{\Sigma}_v$, $\mathbf{s}(t) \in \mathbb{C}^d$ is the transmitted signal, which consists of d independent sources, and $\mathbf{H} = [\mathbf{h}_1, \ldots, \mathbf{h}_d] \in \mathbb{C}^{k \times d}$ is the channel. The temporal correlation of the sources is assumed to follow an exponential-decaying model, namely,

$$[\mathbf{\Sigma}_{s_j}]_{t,t'} = \mathbb{E}[s_j(t)s_j^*(t')] = \gamma_j^{(t-t')},$$

where γ_j is a different constant for each transmitted signal and $j = 1, \ldots, d$. To apply the aforementioned tests, we need to stack q samples of $\mathbf{y}(t)$ to form the observation $\mathbf{x}(t) = [\mathbf{y}^T(tq), \mathbf{y}^T(tq + 1), \ldots, \mathbf{y}^T((t+1)q-1)]^T$.[1] Then, under $\mathcal{H}_0$, the covariance matrix of $\mathbf{x}(t)$ is $\mathbf{\Sigma}_0 = \mathbf{I}_q \otimes \mathbf{\Sigma}_v$ and

$$\mathbf{\Sigma}_1 = \mathbf{I}_q \otimes \mathbf{\Sigma}_v + \sum_{j=1}^{d} \mathbf{\Sigma}_{s_j} \otimes \mathbf{h}_j \mathbf{h}_j^H$$

under $\mathcal{H}_1$. Hence, under $\mathcal{H}_0$ the covariance matrix is block-spherical with block size k, whereas it is block-Toeplitz with block size k under $\mathcal{H}_1$.

In Fig. 1, we plot the receiving operating characteristic (ROC) curves in an experiment with $d = 2$ sources whose SNRs are $[-8, -9]$ dB and $[\gamma_1, \gamma_2] = [0.9, 0.7e^{\imath\pi/4}]$. The dimensional parameters k and q are set as $k = 2$ and $q = 3$, respectively, and a total of $n = 100$ snapshots are collected. The empirical results, as depicted in Fig. 1, show that the iterative-GLRT outperforms the asymptotic-GLRT and LMPIT-inspired test. This is because the detectors from [12] are derived under the assumption that $q \to \infty$, which is far from true in this experiment.

[1] The observations $\mathbf{x}(t)$ will be temporally correlated, but this correlation is ignored for the considered tests.

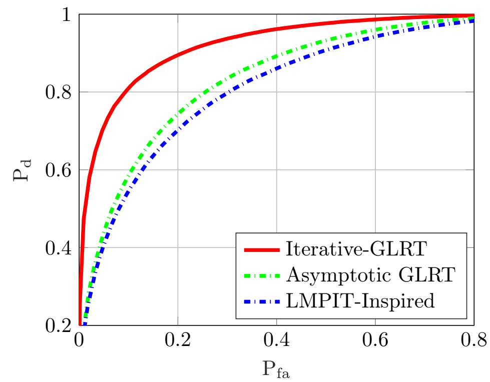

Fig. 1. ROC curves for an experiment with $k = 2$, $q = 3$, $n = 100$ and SNR $= [-8, -9]$ dB

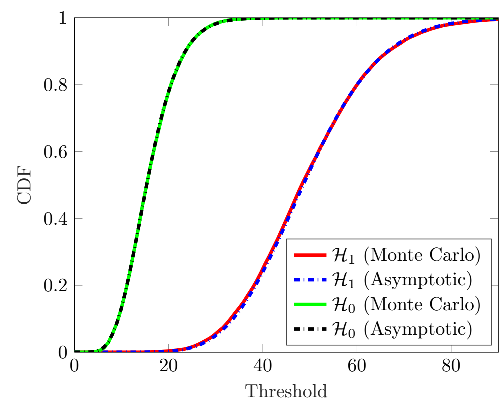

Fig. 2. Accuracy of distributions under the null and the alternative for $k = 2$, $q = 3$, $n = 200$ and snr $= [-8, -9]$ dB

5.3. Theoretical Distribution

Since block-spherical matrices are a special case of block-Toeplitz ones, the asymptotic analytical formulae derived in Section 3.2 are applicable. Thus, in this section we check their accuracy. To do so, we consider an experiment with $n = 200$ observations, while the remaining parameters are those of Fig. 1. The cumulative distribution functions (CDF) are compared to those obtained with Monte Carlo simulations in Fig. 2, where we can observe a very good agreement between the theoretical and experimental results.

6. CONCLUSION

In this paper, we have proposed a general test for the covariance structure of Gaussian populations, which can be applied as long as the structured covariance matrices can be linearly represented by a set of known matrices. The presented approach may be applied to a wide range of detection problems. To solve the detection problem, we resort to the inverse iteration algorithm to seek the ML estimates of the structured covariance matrices, which are required for the GLRT. Moreover, we have also derived the asymptotic distributions under both hypotheses. Looking forward, further studies on the convergence and simplification of the inverse iteration algorithm could prove beneficial.

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