THE RANDOM MONOGENIC SIGNAL

Sofia C. Olhede[*]

Dept. of Statistical Science
University College London, UK
s.olhede@ucl.ac.uk

David Ramírez and Peter J. Schreier[†]

Signal and System Theory Group
Universität Paderborn, Germany
firstname.lastname@sst.upb.de

ABSTRACT

The monogenic signal allows us to decompose a two-dimensional real signal into a local amplitude, a local orientation, and a local phase. In this paper, we introduce the random monogenic signal and study its second-order statistical properties. The monogenic signal may be represented as a quaternion-valued signal. We show that for homogeneous random fields, we need exactly two quaternion-valued covariance functions for a complete second-order description. We also introduce a stochastic model for unidirectional signals and a measure of unidirectionality.

***Index Terms*—** Anisotropy, homogeneous random field, isotropy, monogenic signal, quaternion signals, Riesz transform.

1. INTRODUCTION

There is an infinite number of ways how to decompose a one-dimensional real signal $x(t)$ into a local amplitude $a(t)$ and a local phase $\phi(t)$ as $x(t) = a(t)\cos\phi(t)$. The most meaningful and almost universally accepted way of making this decomposition unique is to work with the analytic signal $x_+(t) = x(t) + i\hat{x}(t)$, where $\hat{x}(t)$ is the Hilbert transform of $x(t)$, and to define $a(t) = |x_+(t)|$ and $\phi(t) = \angle x_+(t)$. The extension to two-dimensional real signals $f(x_1, x_2)$ is a bit more controversial, but many researchers have now accepted the *monogenic signal* [1] as the most appropriate vehicle.

The monogenic signal allows us to decompose a two-dimensional real signal $f(\mathbf{x})$, $\mathbf{x} = [x_1, x_2]^T$, into a local amplitude, a local orientation, and a local phase. This is done by complementing the original signal $f(\mathbf{x})$ by its two Riesz transforms $g(\mathbf{x})$ and $h(\mathbf{x})$ [2]. The monogenic signal can either be represented as a three-dimensional vector $[f(\mathbf{x}), g(\mathbf{x}), h(\mathbf{x})]^T$ or as a quaternion $m(\mathbf{x}) = f(\mathbf{x}) + ig(\mathbf{x}) + jh(\mathbf{x}) + k\cdot 0$, where the k-part remains zero. The monogenic signal has been used for numerous applications in image processing such as improved image estimation [3] and feature extraction [4] . Local monogenic transforms have been constructed to improve the mathematical properties of the decomposition [5, 6].

So far, work on monogenic signals has focussed only on the deterministic case. In this paper, we discuss second-order statistical properties of *random* monogenic signals. As a monogenic signal has three components, we generally need six real-valued covariance functions to account for all possible auto- and cross-covariances between f, g, and h. If we employ the quaternionic description $m(\mathbf{x}) = f(\mathbf{x}) + ig(\mathbf{x}) + jh(\mathbf{x})$, we work with quaternion-valued covariance functions. It is clear, however, that one quaterion-valued covariance function alone cannot contain the same amount of information as six real-valued covariance functions [7]. We will show that, for *homogeneous* random fields, only one further covariance function (a so-called *complementary covariance function*) is needed to fully characterize the second-order statistics of the monogenic signal.

It is commonplace in the analysis of random fields to assume isotropy. Often this is not an unreasonable assumption, but the converse can also be found in applications such as geophysics [8, 9] and medical imaging, where the direction dependence, and the axis of the anisotropy, are important summary statistics. As an application of our second-order theory of monogenic signals, we introduce a stochastic model for unidirectional signals and a measure of unidirectionality. We then present simulation studies for a set of images that show how the degree of anisotropy is reflected in our measure of unidirectionality.

2. THE MONOGENIC SIGNAL

The continuous Riesz transforms [2] of a real-valued function $f(\mathbf{x})$ are given by

$$\mathcal{R}^{(l)} f(\mathbf{x}) \quad = \quad \frac{1}{2\pi} \iint f(\mathbf{y}) \frac{y_l - x_l}{\|\mathbf{x}-\mathbf{y}\|^3}\, d\mathbf{y}, \qquad (1)$$

for $l = 1, 2$ and $\mathbf{x} = [x_1, x_2]^T$, $\mathbf{y} = [y_1, y_2]^T$. The quaternion-valued monogenic signal is formed by placing the Riesz trans-

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forms into i- and j-parts,

$$m(\mathbf{x}) = f(\mathbf{x}) + ig(\mathbf{x}) + jh(\mathbf{x}),$$

with $g(\mathbf{x}) = \mathcal{R}^{(1)} f(\mathbf{x})$ and $h(\mathbf{x}) = \mathcal{R}^{(2)} f(\mathbf{x})$, leaving the k-part empty. For a review of quaternion algebra, see [1].

The monogenic signal can also be constructed in the Fourier domain, as convolutions of the type (1) become multiplications with

$$R^{(l)}(\mathbf{k}) = -ik_l/\|\mathbf{k}\|, \quad \mathbf{k} = [k_1, k_2]^T.$$

If the transformation is implemented over the entire real plane, the space- and wavenumber-representations are equivalent.

3. SECOND ORDER CHARACTERIZATION OF THE MONOGENIC SIGNAL

Let us consider a homogeneous random field $f(\mathbf{x})$, whose covariance function $r_{ff}(\boldsymbol{\xi}) = \operatorname{cov}\{f(\mathbf{x}), f(\mathbf{x}-\boldsymbol{\xi})\}$ is independent of $\mathbf{x}$. It has the spectral representation [10]

$$f(\mathbf{x}) = \iint d\mathcal{Z}_f(\mathbf{k}) e^{2i\pi\mathbf{k}^T\mathbf{x}}.$$

The spectral process $\mathcal{Z}_f(\mathbf{k})$ has uncorrelated increments,

$$\operatorname{cov}\{d\mathcal{Z}_f(\mathbf{k}), d\mathcal{Z}_f(\mathbf{k}')\} = S_{ff}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\, d\mathbf{k}\, d\mathbf{k}',$$

and $S_{ff}(\mathbf{k})$ is the power spectral density of $f(\mathbf{x})$. Using the spectral representation of the Riesz transforms, we may express their auto-covariances as

$$r_{gg}(\boldsymbol{\xi}) = \iint \cos^2(\kappa) S_{ff}(\mathbf{k}) e^{2i\pi\mathbf{k}^T\boldsymbol{\xi}}\, d\mathbf{k},$$
$$r_{hh}(\boldsymbol{\xi}) = \iint \sin^2(\kappa) S_{ff}(\mathbf{k}) e^{2i\pi\mathbf{k}^T\boldsymbol{\xi}}\, d\mathbf{k},$$

where $\kappa = \arg(\mathbf{k})$. Therefore, the covariances are related as $r_{ff}(\boldsymbol{\xi}) = r_{gg}(\boldsymbol{\xi}) + r_{hh}(\boldsymbol{\xi})$. Additionally, we find that the cross-covariances are given by

$$r_{fg}(\boldsymbol{\xi}) = -i \iint \cos(\kappa) S_{ff}(\mathbf{k}) e^{2i\pi\mathbf{k}^T\boldsymbol{\xi}}\, d\mathbf{k},$$
$$r_{fh}(\boldsymbol{\xi}) = -i \iint \sin(\kappa) S_{ff}(\mathbf{k}) e^{2i\pi\mathbf{k}^T\boldsymbol{\xi}}\, d\mathbf{k},$$
$$r_{gh}(\boldsymbol{\xi}) = \frac{1}{2} \iint \sin(2\kappa) S_{ff}(\mathbf{k}) e^{2i\pi\mathbf{k}^T\boldsymbol{\xi}}\, d\mathbf{k},$$

which satisfy $r_{gf}(\boldsymbol{\xi}) = -r_{fg}(\boldsymbol{\xi})$, $r_{hf}(\boldsymbol{\xi}) = -r_{fh}(\boldsymbol{\xi})$, and $r_{hg}(\boldsymbol{\xi}) = r_{gh}(\boldsymbol{\xi})$.

So far we have characterized the monogenic signal in terms of its real components, now we shall obtain the quaternionic characterization. Let us start with the standard quaternion-valued covariance of the monogenic signal, which is

$$\begin{aligned} r_{mm}(\boldsymbol{\xi}) &= \operatorname{cov}\{m(\mathbf{x}), m(\mathbf{x}-\boldsymbol{\xi})\} \\ &= 2r_{ff}(\boldsymbol{\xi}) - 2ir_{fg}(\boldsymbol{\xi}) - 2jr_{fh}(\boldsymbol{\xi}). \end{aligned}$$

As this function only specifies three real covariances, it does not provide a complete second-order description. In order to access the missing information, we need to consider a further *complementary* covariance function. One possible choice is the covariance between $m(\mathbf{x})$ and its involution over i [7], given by $m^{(i)}(\mathbf{x}) = -im(\mathbf{x})i = f(\mathbf{x}) + ig(\mathbf{x}) - jh(\mathbf{x})$, which is

$$r_{mm^{(i)}}(\boldsymbol{\xi}) = 2r_{gg}(\boldsymbol{\xi}) - 2ir_{fg}(\boldsymbol{\xi}) + 2kr_{gh}(\boldsymbol{\xi}).$$

Since $r_{ff}(\boldsymbol{\xi}) - r_{gg}(\boldsymbol{\xi}) = r_{hh}(\boldsymbol{\xi})$, the two quaternion-valued covariances $r_{mm}(\boldsymbol{\xi})$ and $r_{mm^{(i)}}(\boldsymbol{\xi})$ provide the same information as the real-valued covariances.

Instead of $r_{mm^{(i)}}(\boldsymbol{\xi})$, one can also use

$$r_{mm^{(\eta)}}(\boldsymbol{\xi}) = \operatorname{cov}\{m(\mathbf{x}), m^{(\eta)}(\mathbf{x}-\boldsymbol{\xi})\},$$

where η is any pure unit quaternion and $m^{(\eta)}(\mathbf{x}) = -\eta m(\mathbf{x})\eta$ denotes the involution of $m(\mathbf{x})$ over η [7].

However, there are also complementary covariance functions that do not provide the required information. In particular, with $m^*(\mathbf{x}) = f(\mathbf{x}) - ig(\mathbf{x}) - jh(\mathbf{x})$, it is straightforward to show that, for *all homogeneous* random fields, $r_{mm^*}(\boldsymbol{\xi}) = \operatorname{cov}\{m(\mathbf{x}), m^*(\mathbf{x}-\boldsymbol{\xi})\} = 0$.

4. ISOTROPY AND DIRECTIONALITY

We would like to use the monogenic signal to characterize the degree of directionality in a random field. Before we do that, we should clarify what we mean by directionality and lack of directional structure (i.e., isotropy) [11].

Definition 4.1 *A second-order homogeneous random field $f(\mathbf{x})$ is isotropic if the covariance of the field is finite and only depends on the magnitude of the lag, that is if*

$$r_{ff}(\boldsymbol{\xi}) = C_I(\sqrt{\boldsymbol{\xi}^T\boldsymbol{\xi}}),$$

for some square summable function $C_I(\cdot)$.

In statistics, considerable focus is placed on parametric and isotropic random fields [11]. However, in many practical applications data experience strong anisotropy, namely directional dependence. We shall discuss two forms of anisotropic random fields, namely *geometric anisotropy* [12, 13] and *purely directional* signals.

Definition 4.2 *A second-order homogeneous random field $f(\mathbf{x})$ is geometrically anisotropic if the covariance of the field is finite and only depends on the magnitude of the lag once deformed, that is if*

$$r_{ff}(\boldsymbol{\xi}) = C_A(\sqrt{\boldsymbol{\xi}^T\mathbf{D}\boldsymbol{\xi}}),$$

for some 2×2 positive definite symmetric matrix $\mathbf{D}$ with unit determinant and a square summable function $C_A(\cdot)$.

As the condition number of $\mathbf{D}$ increases, the matrix will appear more and more directional. However, no matter what finite condition number is selected, the random field will be an aggregation of more than one wavenumber coefficient. We therefore introduce a third, idealized, class of random field, the *purely directional* signal.

Definition 4.3 *A second-order homogeneous random field $f(\mathbf{x})$ is purely unidirectional if its covariance takes the form*

$$r_{ff}(\boldsymbol{\xi}) = C_U(\mathbf{n}^T\boldsymbol{\xi}),$$

where $\mathbf{n} = [n_1, n_2]^T$ is a unit-norm vector in the direction of the random field and $C_U(\cdot)$ is a square summable function.

A much more interesting way of representing a unidirectional random field is using its spectral representation

$$f(\mathbf{x}) = \int dZ_f(k) e^{2i\pi(k\mathbf{n}^T\mathbf{x}+\phi_0)}, \quad (2)$$

where $\mathrm{cov}\{dZ_f(k), dZ_f(k')\} = S_{ff}(k)\delta(k-k')\,dk\,dk'$. The unidirectional signal is clearly a one-dimensional signal embedded in two dimensions [14]. It is an idealization, just like a perfectly straight infinitely thin line. Geometrically anisotropic signals are more realistic, and allow to represent a continuum of less anisotropic signals, eventually blending into isotropy when $\mathbf{D}$ is the identity matrix.

We now have a hierarchy of signals, starting from isotropic signals (that have no directional preference), to geometrically anisotropic signals (the degree of anisotropy being controlled by the deformation matrix $\mathbf{D}$), ending in purely unidirectional signals.

5. A MEASURE OF UNIDIRECTIONALITY

We are interested in measuring the degree of unidirectionality. Based on its spectral representation (2), we may express the monogenic signal of a unidirectional random field as

$$m(\mathbf{x}) = f(\mathbf{x}) + \eta s(\mathbf{x}),$$

where[1]

$$s(\mathbf{x}) = -i\int \mathrm{sgn}(k)\, dZ_f(k) e^{2i\pi(k\mathbf{n}^T\mathbf{x}+\phi_0)},$$

with direction $\eta = n_1 i + n_2 j$. This yields $m^{(\eta)}(\mathbf{x}) = m(\mathbf{x})$. Therefore, the quantity

$$\min_{\eta} E\left[\frac{1}{2}\left|m(\mathbf{x}) - m^{(\eta)}(\mathbf{x})\right|^2\right] = r_{mm}(\mathbf{0}) - \max_{\eta} \mathrm{Re}\left(r_{mm^{(\eta)}}(\mathbf{0})\right)$$

takes on values between zero and $r_{mm}(\mathbf{0})/2$, where zero indicates a purely unidirectional field. We thus define the *degree of unidirectionality*

$$\mathcal{U} = \frac{\max_{\eta} \mathrm{Re}\left(r_{mm^{(\eta)}}(\mathbf{0})\right)}{r_{mm}(\mathbf{0})}, \quad (3)$$

[1] $s(\mathbf{x})$ may be interpreted as a Hilbert transform along direction η.

which takes on values between $1/2$ and one, and $\mathcal{U} = 1$ for a purely unidirectional signal. After some tedious algebra, we may express (3) as

$$\mathcal{U} = \frac{\lambda_{\max}(\mathbf{R})}{r_{mm}(\mathbf{0})},$$

where $\lambda_{\max}(\mathbf{R})$ is the largest eigenvalue of

$$\mathbf{R} = \begin{bmatrix} \mathrm{Re}(r_{mm^{(i)}}(\mathbf{0})) & \mathrm{Im}_k(r_{mm^{(i)}}(\mathbf{0})) \\ -\mathrm{Im}_k(r_{mm^{(j)}}(\mathbf{0})) & \mathrm{Re}(r_{mm^{(j)}}(\mathbf{0})) \end{bmatrix},$$

and $\mathrm{Im}_k(\cdot)$ denotes the k-part of a quaternion. Moreover, the direction is given by the dominant eigenvector of $\mathbf{R}$.

6. NUMERICAL RESULTS

In this section we present some simulation results to illustrate the behavior of $\mathcal{U}$ for the three different types of random fields presented in Section 4. We generate isotropic and geometric anisotropic random fields using a Matérn covariance function [11] of

$$S_{ff}(\mathbf{k}) = \frac{\sigma^2\Gamma(\nu+1)(4\nu)^\nu}{\pi\Gamma(\nu)(\pi\rho)^{2\nu}\left[4\nu/(\pi\rho)^2 + \mathbf{k}^T\mathbf{D}\mathbf{k}\right]^{\nu+1}},$$

where $\Gamma(\cdot)$ denotes the Gamma-function. We have chosen $\nu = 1.5$, $\rho = 16$ and $\sigma^2 = 1$, and $\mathbf{D} = \mathbf{I}_2$ for the isotropic field, and

$$\mathbf{D} = \frac{1}{\sqrt{0.0975}}\begin{bmatrix} 1 & -0.95 \\ -0.95 & 1 \end{bmatrix},$$

for the anisotropic random field. We have employed the "Approximate Frequency Domain Method" [15], which produces a random field with uncorrelated discrete Fourier coefficients with variance equal to the spectral density. The purely unidirectional signal has spectrum

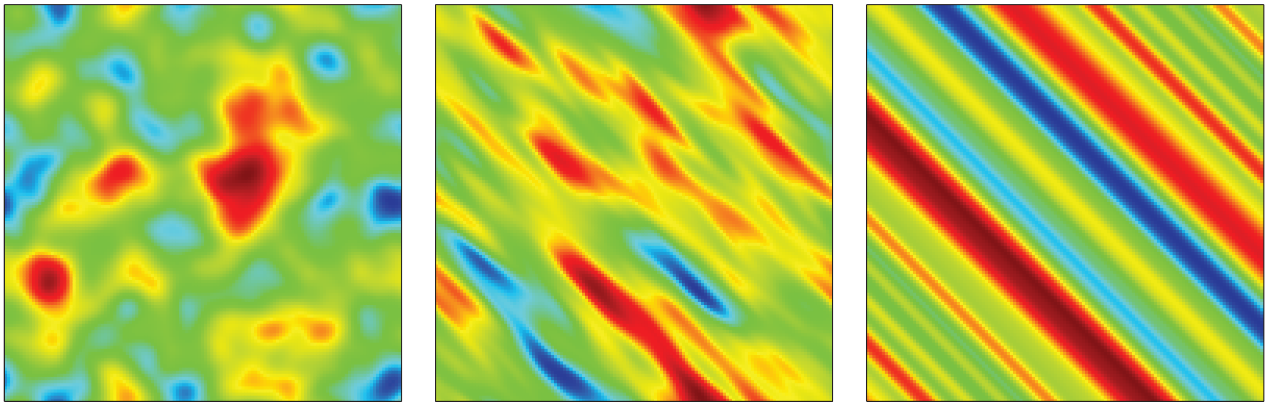
$$S_{ff}(k) = \frac{\sigma^2\Gamma(\nu+1)(4\nu)^\nu}{\pi\Gamma(\nu)(\pi\rho)^{2\nu}\left[4\nu/(\pi\rho)^2 + Dk^2\right]^{\nu+1}},$$

along the line $\mathbf{k} = k\mathbf{n}$ with $\nu = 3, \rho = 2, \sigma^2 = 1$ and $D = 12$.

Figure 1 shows sample images of size 128×128 of these three random fields. Table 1 gives mean values and standard deviations for $\mathcal{U}$, computed from 1000 realizations. We can see that more directional random fields have higher values of $\mathcal{U}$. The reason that the unidirectional random field does not have unit degree of unidirectionality is that we work with sampled random fields.

Isotropic	Anisotropic	Unidirectional
0.5742 ± 0.0381	0.8287 ± 0.0373	0.9188 ± 0.0475

Table 1: Mean values and standard deviations for the degree of unidirectionality for different types of random fields.



(a) Isotropic (b) Anisotropic (c) Unidirectional

Fig. 1: Sample images of random fields

7. CONCLUSIONS

We have introduced the random monogenic signal and studied its second-order statistical properties. We have shown that for homogeneous random fields, we need exactly two quaternion-valued covariance functions for a complete second-order description. We have also introduced a stochastic model for unidirectional signals and a measure of unidirectionality. Our measure can be used to automatically classify images based on their degree of anisotropy, without assuming a parametric form of the anisotropy such as a Matérn class model. More details, including the distributional properties of the degree of unidirectionality, can be found in a forthcoming journal paper.

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