The Locally Most Powerful Invariant Test for Detecting a Rank-P Gaussian Signal in White Noise

David Ramírez
Dept. of Electrical Engineering and Information Technology
Universität Paderborn
Paderborn, Germany
Email: david.ramirez@sst.upb.de

Jorge Iscar, Javier Vía and Ignacio Santamaria
Dept. of Communications Engineering
University of Cantabria
Santander, Spain
Email: {jorge, jvia, nacho}@gtas.dicom.unican.es

Louis L. Scharf
Depts. of Mathematics and Statistics
Colorado State University
Fort Collins, USA
Email: scharf@engr.colostate.edu

***Abstract*—Spectrum sensing has become one of the main components of a cognitive transmitter. Conventional detectors suffer from noise power uncertainties and multiantenna detectors have been proposed to overcome this difficulty, and to improve the detection performance. However, most of the proposed multiantenna detectors are based on non-optimal techniques, such as the generalized likelihood ratio test (GLRT), or even heuristic approaches that are not based on first principles. In this work, we derive the locally most powerful invariant test (LMPIT), that is, the optimal invariant detector for close hypotheses, or equivalently, for a low signal-to-noise ratio (SNR). The traditional approach, based on the distributions of the maximal invariant statistic, is avoided thanks to Wijsman's theorem, which does not need these distributions. Our findings show that, in the low SNR regime, and in contrast to the GLRT, the additional spatial structure imposed by the signal model is irrelevant for optimal detection. Finally, we use Monte Carlo simulations to illustrate the good performance of the LMPIT.**

***Index Terms*—Cognitive Radio, locally most powerful invariant test (LMPIT), multi antenna spectrum sensing, Wijsman's theorem.**

I. Introduction

The cognitive radio (CR) paradigm is a new technology that aims to improve spectrum usage and alleviate the apparent scarcity of resources [1]. The main idea is based on the observation that there are temporally and/or geographically unused bands. Basically, CR relies on the opportunistic access of non-licensed users (secondary users) to these unused bands, avoiding interfering with rightful license owners. This translates into the need for powerful spectrum sensing (detection of primary activity) in CR networks. However, as might be expected, this is a challenging task due to shadowing and fading phenomena, as well as to the low SNR conditions.

To obtain the desired performance in such challenging environments, spectrum sensing techniques may exploit certain features of the signals, such as cyclostationarity and/or the presence of pilots. However, those approaches usually require a level of synchronization not achievable in most practical scenarios. Hence, asynchronous detectors are of interest, the most popular being the energy detector (ED). Nevertheless, in the presence of noise power uncertainties, the performance of the energy detector is severely degraded [2]. To overcome this problem, and improve the detection performance, multiantenna detectors are typically considered. These detectors are usually based on the generalized likelihood ratio test (see [3] and references therein), or heuristic approaches. However, it is well known that, in general, these techniques lead to suboptimal tests.

In this work, we derive the locally most powerful invariant test (LMPIT) for the typical signal-plus-noise model, usually found in communication systems. Hence, we obtain the optimal invariant detector in the low signal-to-noise ratio (SNR) regime. The derivation of the LMPIT usually becomes a complicated task because we need to derive the distributions of the maximal invariant. However, using Wijsman's theorem [4], [5], we can calculate the ratio of the distributions of the maximal invariant statistic without deriving the distributions, and even without obtaining the maximal invariant statistic. On the other hand, we must point out that the LMPIT for a closely related problem was obtained in [6], where the author used the (more complicated) conventional approach for deriving the LMPIT.

II. System Model

We consider the problem of spectrum sensing using multiantenna spectrum monitors. The problem is formulated in such a way that it does not require prior knowledge about the primary signals, the wireless channel, nor the noise processes (beyond spatial independence). We consider P signals, which may be generated by a P-antenna transmitter, P single-antenna users, or any combination, impinging on an L-antenna spectrum monitor. At the receiver, the signals are downconverted and sampled at the Nyquist rate, without synchronization with any potentially present primary signal. Taking this into account, the spectrum sensing problem is formulated as

$$\begin{aligned} &\mathcal{H}_1 : \mathbf{x}[n] = \mathbf{H}\mathbf{s}[n] + \mathbf{v}[n], \quad && n = 0, \ldots, M-1 \\ &\mathcal{H}_0 : \mathbf{x}[n] = \mathbf{v}[n], && n = 0, \ldots, M-1 \end{aligned} \tag{1}$$

where $\mathbf{s}[n] = [s_1[n], \ldots, s_P[n]]^T$ is the transmitted signal, $\mathbf{H} \in \mathbb{C}^{L \times P}$ is the wireless channel, and $\mathbf{v}[n] = [v_1[n], \ldots, v_L[n]]^T$ is the additive noise vector, which is assumed to be zero-mean circular complex Gaussian, independent of $\mathbf{s}[n]$, and spatially and temporally white. Moreover, the transmitted signal is considered temporally and spatially white, and the channel is flat fading. Under these assumptions, the covariance matrix of the measurements is, for all n,

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{R} = \mathbf{H}\mathbf{H}^H + \sigma^2 \mathbf{I}, \\ \mathcal{H}_0 &: \mathbf{R} = \sigma^2 \mathbf{I}, \end{aligned} \tag{2}$$

where σ^2 is the noise power and $\mathbf{R} = E[\mathbf{x}[n]\mathbf{x}^H[n]]$.

Before proceeding, we assume that $\mathbf{s}[n]$ is zero-mean, circular complex Gaussian, which will result in tractable analysis and useful detectors, and can also be seen as a worst case distribution [7]. Therefore, the detection problem in (2) becomes

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{x}[n] \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{H}\mathbf{H}^H + \sigma^2 \mathbf{I}\right), \\ \mathcal{H}_0 &: \mathbf{x}[n] \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right), \end{aligned} \tag{3}$$

where $\mathcal{CN}(\boldsymbol{\mu}, \mathbf{R})$ stands for the complex circular Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{R}$. Hence, the spectrum sensing problem has been formulated as a test for the covariance structure of the received measurements. Specifically, under $\mathcal{H}_0$ the covariance matrix is proportional to the identity matrix, whereas, under $\mathcal{H}_1$, it is the sum of a rank-P matrix and the noise covariance matrix.

III. Previous work

A. Generalized Likelihood Ratio Test (GLRT)

The detection problem in (3) is a composite test, that is, the likelihoods depend on unknown parameters. Previous approaches to this problem in the context of spectrum sensing are based on the well-known generalized likelihood ratio test (GLRT), which, basically, obtains the likelihood ratio with the unknown parameters replaced by their maximum likelihood (ML) estimates. In particular, for the test given in (3), and considering M independent and identically distributed (i.i.d.) samples of $\mathbf{x}[n]$, the generalized likelihood ratio (GLR) is given by

$$\mathscr{G} = \frac{\max\limits_{\sigma^2 > 0} p\left(\mathbf{X}; \sigma^2 \mathbf{I}\right)}{\max\limits_{\substack{\mathbf{R}_1 = \mathbf{H}\mathbf{H}^H + \sigma^2 \mathbf{I}, \\ \sigma^2 > 0}} p\left(\mathbf{X}; \mathbf{R}_1\right)},$$

where the likelihood is

$$p\left(\mathbf{X}; \mathbf{R}\right) = \frac{1}{\pi^{LM} |\mathbf{R}|^M} \exp\left[-M \operatorname{tr}\left(\mathbf{R}^{-1}\hat{\mathbf{R}}\right)\right],$$

with $|\cdot|$ and $\operatorname{tr}(\cdot)$ denoting the determinant and the trace of a matrix, respectively. The data matrix is given by $\mathbf{X} = \left[\mathbf{x}[0], \ldots, \mathbf{x}[M-1]\right]$ and $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H/M$ is the sample covariance matrix. Finally, the test is obtained by comparing the GLR with a threshold. In [3], it was shown that the GLRT depends on the rank P. For $P \geq L-1$, the test becomes the well known test for sphericity, and the GLRT is given by [8]

$$\log \mathscr{G} = \log \left[\frac{\left(\prod_{i=1}^{L} \lambda_i\right)^{1/L}}{\frac{1}{L}\sum_{i=1}^{L} \lambda_i} \right] \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \eta,$$

where λ_i is the i-th eigenvalue of $\hat{\mathbf{R}}$. On the contrary, for $P < L-1$, the GLRT becomes [3]

$$\begin{aligned} \log \mathscr{G} = {} & L \log \left[\frac{\left(\prod_{i=1}^{L} \lambda_i\right)^{1/L}}{\frac{1}{L}\sum_{i=1}^{L} \lambda_i} \right] \\ & - (L-P) \log \left[\frac{\left(\prod_{i=P+1}^{L} \lambda_i\right)^{1/(L-P)}}{\frac{1}{L-P}\sum_{i=P+1}^{L} \lambda_i} \right] \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \eta. \end{aligned}$$

B. Locally Most Powerful Invariant Test (LMPIT)

Despite its simplicity, the GLRT does not have any optimality property and, in fact, its performance might be poor when the sample size is small. Therefore, other detectors have been considered for dealing with composite hypotheses. The optimal test (uniformly most powerful test) does not exist in general, because the likelihood ratio depends on the unknown parameters. One possible way to avoid this dependence is restricting ourselves to the class of invariant detectors [9]. In this way we obtain the optimal test among those invariant, that is, the uniformly most powerful invariant test (UMPIT). However, when the UMPIT does not exist, we can still derive an optimal test, under the assumption of close hypotheses, i.e., the locally most powerful invariant test (LMPIT).

The typical approach for deriving the LMPIT is [9]: (i) identify the invariances of the problem and describe them as a group of transformations, (ii) find the maximal invariant statistic, (iii) derive the distribution of the maximal invariant statistic under both hypotheses, (iv) calculate the ratio of the distributions of the maximal invariant statistic, and (v) apply a Taylor's expansion of the likelihood ratio.[1] This approach was used in [6] to derive the LMPIT for the sphericity test

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{x}[n] \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{R}\right), \\ \mathcal{H}_0 &: \mathbf{x}[n] \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right), \end{aligned}$$

with $\mathbf{R}$ only constrained to be a positive definite matrix. In particular, the LMPIT is given by

$$\mathscr{L} = \|\hat{\hat{\mathbf{R}}}\|_F^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta, \tag{4}$$

where

$$\hat{\hat{\mathbf{R}}} = \frac{\hat{\mathbf{R}}}{\operatorname{tr}\left(\hat{\mathbf{R}}\right)}. \tag{5}$$

[1] Step (v) is only necessary if the likelihood ratio depends on the unknown parameters. Otherwise, the density ratio of the maximal invariants would provide the uniformly most powerful invariant test (UMPIT) statistic.

IV. Derivation of the LMPIT: Wijsman's Theorem

The approach for deriving the LMPIT discussed in Section III-B might be quite complicated, not only due to the need of obtaining the distributions of the maximal invariant statistic, but also due to the need of an explicit formulation for the maximal invariant. Here, we use another approach, namely Wijsman's theorem [4], [5]. This theorem allows us to directly obtain the maximal invariant density ratio without knowledge of the distributions of the maximal invariant statistic, and even without an explicit expression for the maximal invariant statistic. Wijsman's theorem states that, under some mild conditions, the likelihood ratio of the maximal invariant statistic is given by

$$\mathscr{L} = \frac{\int_{\mathcal{G}} p\left(g(\mathbf{y}); \mathcal{H}_1\right) |\mathbf{J}_g| \, dg}{\int_{\mathcal{G}} p\left(g(\mathbf{y}); \mathcal{H}_0\right) |\mathbf{J}_g| \, dg},$$

where $\mathcal{G}$ is the group of transformations under which the test is invariant, $|\mathbf{J}_g|$ denotes the absolute value of the Jacobian of the transformations $g(\cdot) \in \mathcal{G}$, and dg is an invariant group measure.

To apply Wijsman's theorem to our problem, the first step is to identify the problem invariances. Concretely, the group of invariant transformations for the detection problem given in (3) is

$$\mathcal{G} = \left\{ g : \mathbf{x}[n] \rightarrow g(\mathbf{x}[n]) = a\mathbf{Q}\mathbf{x}[n] \right\},$$

where $a \in \mathbb{R}_+$ is a positive real number, and $\mathbf{Q} \in \mathbb{U}$, with $\mathbb{U}$ denoting the set of unitary matrices. This means that scaling all received signals by a common factor, or multiplying them by a unitary matrix should not change the test. Taking into account the invariances of our problem, Wijsman's theorem reads as follows

$$\mathscr{L} = \frac{\int_{\mathbb{R}_+} \int_{\mathbb{U}} |\mathbf{S}|^{2M} a^{2M} e^{-a^2 M \operatorname{tr}\left(\mathbf{S}\mathbf{Q}\hat{\mathbf{R}}\mathbf{Q}^H\right)} d\mathbf{Q} da}{\int_{\mathbb{R}_+} \int_{\mathbb{U}} \sigma^{-2ML} a^{2M} e^{-\frac{a^2}{\sigma^2} M \operatorname{tr}\left(\hat{\mathbf{R}}\right)} d\mathbf{Q} da},$$

where $\mathbf{S} = \left(\mathbf{H}\mathbf{H}^H + \sigma^2 \mathbf{I}\right)^{-1}$. Now, to obtain the LMPIT, we have to decompose $\mathscr{L}$ as a function only dependent on the data and a term not dependent on the data. Before proceeding, let us present the following lemma.

Lemma 1: The ratio $\mathscr{L}$ is given (up to additive and multiplicative constant terms not depending on data) by

$$\mathscr{L} \propto \int_{\mathbb{R}_+} \int_{\mathbb{U}} a^{2M} \mathrm{e}^{-\alpha} d\mathbf{Q} da, \tag{6}$$

where

$$\alpha = a^2 M \sum_{k,m=1}^{L} \lambda_k |q_{m,k}|^2 \beta_m.$$

Here λ_k and β_m denote the eigenvalues of $\hat{\hat{\mathbf{R}}}$ and $\mathbf{S}$, and $q_{m,k}$ is the $\{m,k\}$-th element of $\mathbf{Q}$.

Proof: The proof follows directly by simply applying the following changes of variable:

$$a^2 \rightarrow a^2 / \operatorname{tr}\left(\hat{\mathbf{R}}\right), \qquad \mathbf{Q} \rightarrow \mathbf{U}_{\mathbf{S}} \mathbf{Q} \mathbf{U}_{\hat{\mathbf{R}}}^H,$$

where $\mathbf{U}_{\mathbf{S}}$ and $\mathbf{U}_{\hat{\mathbf{R}}}$ are the matrices of eigenvectors of $\mathbf{S}$ and $\hat{\mathbf{R}}$, respectively. ∎

It is easy to check that (6) is a function of the unknown parameters, thus proving that the UMPIT does not exist, in general. The only exception is the case of $L = 2$ antennas, where the test statistic of the UMPIT [10] is given by the largest eigenvalue of $\hat{\hat{\mathbf{R}}}$, i.e., λ_1. Due to the non-existence of the UMPIT, we consider the challenging low-SNR scenario and derive the LMPIT. Note that in the low-SNR regime $\mathbf{S} \approx \sigma^{-2}\mathbf{I} \Rightarrow \alpha_0 = a^2 M/\sigma^2$, and applying a second order Taylor series expansion of $\mathrm{e}^{-\alpha}$, we get

$$\mathscr{L} \propto \int_{\mathbb{R}_+} \int_{\mathbb{U}} a^{2M} \mathrm{e}^{-\alpha_0} \left[(\alpha - \alpha_0)^2 - (\alpha - \alpha_0) \right] d\mathbf{Q} da,$$

Finally, the LMPIT is presented in the following result.

Theorem 1: The LMPIT is given by

$$\mathscr{L} \propto \sum_{k=1}^{L} \lambda_k^2 = \|\hat{\hat{\mathbf{R}}}\|_{\mathrm{F}}^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta. \tag{7}$$

Proof: Here, due to the lack of space, we only present a sketch of the proof. First, let us consider the linear term, which, neglecting the sign, may be rewritten as

$$\Delta \sum_{k,m=1}^{L} \lambda_k \beta_m = \Delta \operatorname{tr}\left(\hat{\hat{\mathbf{R}}}\right) \operatorname{tr}\left(\mathbf{S}\right) = \Delta \operatorname{tr}\left(\mathbf{S}\right),$$

where

$$\Delta = \int_{\mathbb{R}_+} \int_{\mathbb{U}} a^{2M+1} \mathrm{e}^{-a^2 M/\sigma^2} |q_{m,k}|^2 d\mathbf{Q} da,$$

and we have taken into account that the value of the integral does not depend on the actual indexes. Hence, we can notice that the linear term does not depend on the data, so it can be neglected. The quadratic term, after expanding the square, becomes

$$\sum_{k,m,l,s} \lambda_k \lambda_l \beta_m \beta_s \int_{\mathbb{R}_+} \int_{\mathbb{U}} a^{2M+1} \mathrm{e}^{-a^2 M/\sigma^2} |q_{m,k}|^2 |q_{s,l}|^2 d\mathbf{Q} da.$$

Now, decomposing the above summations and taking into account that the value of the integral is the same for some subsets of the indexes, the proof follows. ∎

A. Further Comments

This work has derived the LMPIT using a (in our opinion) more powerful approach, Wijsman's theorem, whereas [6] uses the conventional approach, requiring an explicit formulation of the maximal invariant statistic and its distributions. Moreover, [6] did not consider the additional spatial structure imposed by the signal-plus-noise model. From our results, we can conclude that in the low-SNR regime the additional structure imposed by the signal model given in (1) and the hypothesis test in (3) is irrelevant for optimal detection. On the contrary, it is known that the GLRT depends on the spatial structure of the measurements under $\mathcal{H}_1$.

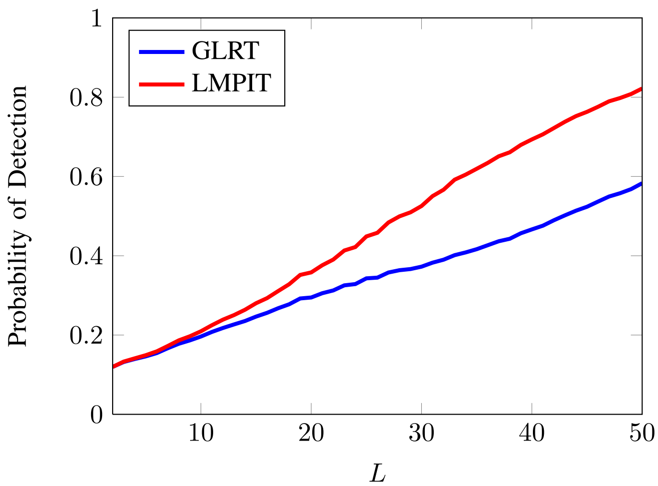

Fig. 1. Probability of detection vs. L for $P = 30$, $M = 50$, $p_{fa} = 0.1$ and SNR $= -6$ dB.

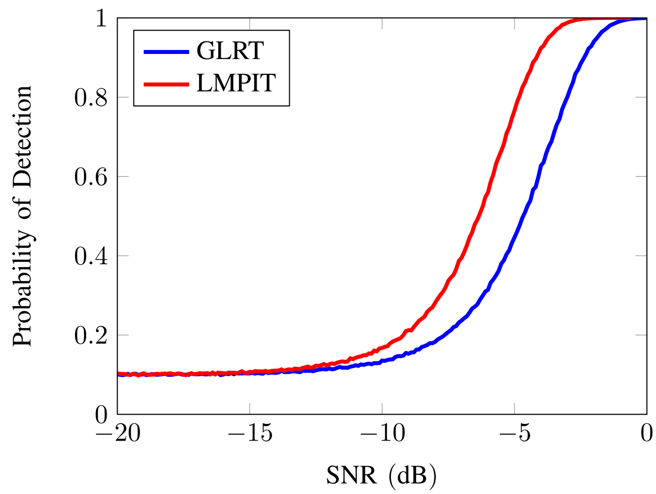

Fig. 2. Probability of detection for $p_{fa} = 0.1$, $L = 40$, $P = 30$ and $M = 40$.

V. Numerical Results

In this section we evaluate the performance of the LMPIT by means of Monte Carlo simulations. The noise power σ^2 is fixed during the experiment, and the entries of the channel matrix are i.i.d. complex Gaussian random variables scaled to achieve the same SNR per realization, with the SNR defined as follows

$$\text{SNR (dB)} = 10 \log_{10} \frac{\operatorname{tr}\left(\mathbf{H}\mathbf{H}^H\right)}{L\sigma^2}.$$

In the first experiment, we obtain the probability of detection for a fixed false alarm probability ($p_{fa} = 0.1$) and varying L, the remaining parameters are $P = 30$, $M = 50$ samples and SNR $= -6$ dB. Figure 1 shows that the performance of the LMPIT is better than that of the GLRT, and these differences are greater for large values[2] of L.

The second experiment analyzes the performance of the GLRT and the LMPIT for different values of the SNR. Figure 2 shows the probability of detection for a fixed probability of false alarm $p_{fa} = 0.1$, $L = 40$, $P = 30$ and $M = 40$. As expected, the performance of the LMPIT is better for these small to moderate values of the SNR.

VI. Conclusions

We have derived the locally most powerful invariant test (LMPIT) for a spectrum sensing problem in cognitive radio. That is, we have obtained the optimal invariant test in the low SNR regime. Interestingly, it turns out that the LMPIT coincides with a previously proposed LMPIT for a closely related problem, where the covariance matrix under the alternative is only constrained to be positive definite. Hence, this work has proved that the actual spatial structure of the covariance matrix under the true hypothesis is irrelevant for optimal detection in the low-SNR region, contrary to the GLRT, which tries to exploit that information. Additionally, we have followed an alternative approach for deriving the LMPIT, based on Wijsman's theorem, which overcomes the need for deriving the distributions of the maximal invariant statistic. Finally, as expected, simulation results illustrate the better performance of the LMPIT for small to moderate values of the SNR.

[2]In a practical scenario, it is impossible to have a spectrum monitor with such a large number of antennas, but it might be possible to use several cooperative spectral monitors to achieve such a large number.

Acknowledgments

The work of J. Iscar, J. Vía and I. Santamaría was supported by the Spanish Government, Ministerio de Ciencia e Innovación (MICINN), under projects COSIMA (TEC2010-19545-C04-03) and COMONSENS (CSD2008-00010, CONSOLIDER-INGENIO 2010 Program). The work of L. L. Scharf was supported by the Airforce Office of Scientific Research under contract FA9550-10-1-0241.

References

[1] J. Mitola and G. Q. Maguire Jr., "Cognitive radio: Making software radios more personal," *IEEE Pers. Comm.*, vol. 6, pp. 13–18, Aug. 1999.

[2] R. Tandra and A. Sahai, "SNR walls for signal detection," *IEEE J. Sel. Topics Signal Process.*, vol. 2, no. 1, pp. 4–17, Feb. 2008.

[3] D. Ramírez, G. Vázquez-Vilar, R. López-Valcarce, J. Vía, and I. Santamaría, "Multiantenna detection of rank-P signals in cognitive radio networks," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3764–3774, Aug. 2011.

[4] R. A. Wijsman, "Cross-sections of orbits and their application to densities of maximal invariants," in *Proc. Fifth Berkeley Symp. on Math. Stat. and Prob.*, vol. 1, 1967, pp. 389–400.

[5] J. Gabriel and S. Kay, "Use of Wijsman's theorem for the ratio of maximal invariant densities in signal detection applications," in *Asilomar Conf. Signals, Systems and Computers*, vol. 1, Nov. 2002, pp. 756 – 762.

[6] S. John, "Some optimal multivariate tests," *Biometrika*, vol. 58, no. 1, pp. 123–127, 1971.

[7] J. Villares and G. Vázquez, "The Gaussian assumption in second-order estimation problems in digital communications," *IEEE Trans. Signal Process.*, vol. 55, no. 10, pp. 4994–5002, 2007.

[8] J. Mauchly, "Significance test for sphericity of a normal n-variate distribution," *Ann. Math. Statist.*, vol. 11, pp. 204–209, 1940.

[9] L. L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*. Addison - Wesley, 1991.

[10] O. Besson, S. Kraut, and L. L. Scharf, "Detection of an unknown rank-one component in white noise," *IEEE Trans. Signal Process.*, vol. 54, no. 7, pp. 2835–2839, Jul. 2006.