THE LOCALLY MOST POWERFUL TEST FOR MULTIANTENNA SPECTRUM SENSING WITH UNCALIBRATED RECEIVERS

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ABSTRACT

Spectrum sensing is a key component of the cognitive radio (CR) paradigm. Among CR detectors, multiantenna detectors are gaining popularity since they improve the detection performance and are robust to noise uncertainties. Traditional approaches to multiantenna spectrum sensing are based on the generalized likelihood ratio test (GLRT) or other heuristic detectors, which are not optimal in the Neyman-Pearson sense. In this work, we derive the locally most powerful invariant test (LMPIT), which is the optimal detector, among those preserving the problem invariances, in the low SNR regime. In particular, we apply Wijsman's theorem, which provides us an alternative way to derive the ratio of the distributions of the maximal invariant statistic. Finally, numerical simulations illustrate the performance of the proposed detector.

***Index Terms*—** Cognitive radio (CR), generalized likelihood ratio test (GLRT), locally most powerful invariant test (LMPIT), multiantenna spectrum sensing.

1. INTRODUCTION

Cognitive radio (CR) paradigm has emerged as a new technology to improve the spectrum usage and alleviate the apparent scarcity of resources [1]. The main idea is to allow the opportunistic transmission of non-licensed users (secondary users) to temporally and/or geographically unused bands avoiding interfering the rightful license owners. Hence, spectrum sensing (detection of primary activity) is a key component of CR networks. Nevertheless, this is a challenging task due to the shadowing and fading phenomena, as well as to the low SNR conditions.

Spectrum sensing techniques may exploit certain features of the signals, but those approaches usually require a level of synchronization not achievable in most practical scenarios. Hence, asynchronous detectors are of interest. The most popular asynchronous spectrum sensing technique is given by the energy detector (ED), whose performance is seriously degraded in presence of noise variance uncertainties [2]. To overcome this problem and improve the detection performance, multiantenna detectors may be considered, which are frequently obtained by deriving the generalized likelihood ratio test (see [3, 4] and references therein) or other heuristic approaches. However, none of such techniques results, in general, in optimal tests.

In this work, we derive the locally most powerful invariant test (LMPIT) for the problem of testing whether a set of measurements are spatially correlated or not. That is, we obtain the optimal invariant detector for close hypotheses, or equivalently, for low signal to noise ratios (SNRs). This is a complicated task since we have to obtain the distribution, under each hypothesis, of the maximal invariant statistic. Nevertheless, this may be overcome using Wijsman's theorem [5, 6], which allows us to calculate the ratio of the distributions of the maximal invariant statistic without deriving the distributions, and even without obtaining the maximal invariant statistic.

2. PROBLEM FORMULATION

In this section we address the problem of spectrum sensing for cognitive radio networks using spectral monitors equipped with L antennas. Specifically, we formulate the problem as a hypothesis test which requires no prior knowledge about the primary signals, the wireless channel, nor the noise processes (beyond spatial independence). The received signals are downconverted and sampled at the Nyquist rate, assuming no synchronization with any potentially present primary signal. Hence, the spectrum sensing problem is formulated as the following hypothesis test

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{x}[n] = \mathbf{s}[n] + \mathbf{v}[n], \\ \mathcal{H}_0 &: \mathbf{x}[n] = \mathbf{v}[n], \end{aligned} \tag{1}$$

where $\mathbf{s}[n] = [s_1[n], \ldots, s_L[n]]^T$ is the temporally white signal received at the L antennas, and $\mathbf{v}[n] = [v_1[n], \ldots, v_L[n]]^T$ is the additive noise vector, which is assumed to be zero-mean circular complex Gaussian, independent of $\mathbf{s}[n]$, and spatially and temporally white. Under these assumptions, the covariance matrices of the signal and noise are

$$E[\mathbf{s}[n]\mathbf{s}^H[n]] = \mathbf{R}_s, \qquad E[\mathbf{v}[n]\mathbf{v}^H[n]] = \mathbf{D},$$

where $\mathbf{R}_s$ is any positive semi-definite matrix and $\mathbf{D}$ is a diagonal matrix with positive entries. Moreover, we suppose that the sum $\mathbf{R}_s + \mathbf{D}$ is a positive definite matrix without further structure.

In order to proceed, we assume that $\mathbf{s}[n]$ is zero-mean, circular complex Gaussian, which will result in tractable analysis and useful detectors, and can also be seen as a worst case [7] distribution. Therefore, the detection problem in (1) becomes

$$\begin{aligned} \mathcal{H}_1 &: \mathbf{x}[n] \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_1), \\ \mathcal{H}_0 &: \mathbf{x}[n] \sim \mathcal{CN}(\mathbf{0}, \mathbf{D}), \end{aligned} \tag{2}$$

where $\mathcal{CN}(\boldsymbol{\mu}, \mathbf{R})$ stands for the complex circular Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{R}$, and $\mathbf{R}_1 = \mathbf{R}_s + \mathbf{D}$ is any positive definite matrix. Hence, under $\mathcal{H}_0$ the covariance matrix is diagonal, whereas it has no additional structure under $\mathcal{H}_1$, beyond being positive definite. To summarize, the spectrum sensing problem has been formulated as a test for the covariance structure of the Gaussian vector $\mathbf{x}[n]$.

3. GENERALIZED LIKELIHOOD RATIO TEST

Let us start by briefly reviewing the generalized likelihood ratio test (GLRT) [4, 8] for the model (2), which will also allow us to introduce some important definitions. We shall consider an experiment producing $M \geq L$ independent and identically distributed (iid) realizations of $\mathbf{x}[n]$. Therefore, the likelihood is given by the product of the individual pdfs, yielding

$$p(\mathbf{X}; \mathbf{R}) = \frac{1}{\pi^{LM} |\mathbf{R}|^M} \exp\left[-M \operatorname{tr}\left(\mathbf{R}^{-1}\hat{\mathbf{R}}\right)\right],$$

where $|\cdot|$ and $\operatorname{tr}(\cdot)$ denote the determinant and the trace of a matrix, respectively. The data matrix is given by $\mathbf{X} = \left[\mathbf{x}[0], \ldots, \mathbf{x}[M-1]\right]$ and $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H/M$, is the sample covariance matrix.

The generalized likelihood ratio (GLR) for the test (2) is given by

$$\mathscr{G} = \frac{\max_{\mathbf{D} \in \mathbb{D}_+} p(\mathbf{X}; \mathbf{D})}{\max_{\mathbf{R}_1 \in \mathbb{S}} p(\mathbf{X}; \mathbf{R}_1)},$$

where $\mathbb{D}_+$ is the set of diagonal matrices with positive entries and $\mathbb{S}$ is the set of positive definite matrices without further structure. To find the GLRT we have to obtain the maximum likelihood estimates of $\mathbf{R}_1$ and $\mathbf{D}$, and plug them back into the GLR. Finally, the test is obtained by comparing the GLR with a threshold.

The ML estimates are given by $\hat{\mathbf{R}}_1 = \hat{\mathbf{R}}$ and

$$\hat{\mathbf{D}} = \operatorname{diag}(\hat{r}_{1,1}, \ldots, \hat{r}_{L,L}),$$

and taking them into account, the GLRT is given by

$$\mathscr{G} = \frac{|\hat{\mathbf{R}}|}{\prod_{k=1}^{L} \hat{r}_{k,k}} = |\hat{\mathbf{C}}| \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \eta_{\mathrm{G}}, \tag{3}$$

where $\hat{\mathbf{C}} = \hat{\mathbf{D}}^{-1/2}\hat{\mathbf{R}}\hat{\mathbf{D}}^{-1/2}$ is the sample coherence matrix and η_{G} a properly selected threshold.

4. LOCALLY MOST POWERFUL INVARIANT TEST

In this section we derive the locally most powerful invariant test (LMPIT) for the detection problem in (2). Typically, the LMPIT is obtained as follows [9]: (i) identify the invariances of the problem and describe them as a group of transformations, (ii) find the maximal invariant statistic, (iii) derive the distribution of the maximal invariant statistic under both hypotheses, (iv) calculate the ratio of the distributions of the maximal invariant statistic, and (v) apply a Taylor's expansion of the likelihood ratio.[1]

The above problem is, in general, very difficult, which is not only due to the need of obtaining the distributions of the maximal invariant statistic, but also due to the need of an explicit formulation for the maximal invariant statistic. An alternative way to derive the LMPIT that circumvents this problem is based on Wijsman's theorem [5, 6], which allows us to directly obtain the maximal invariant density ratio even without an explicit expression for the maximal invariant statistic. In particular, this theorem states that, under some mild conditions, the likelihood ratio of the maximal invariant statistic is given by

$$\mathscr{L} = \frac{\int_{\mathcal{G}} p(g(\mathbf{y}); \mathcal{H}_1) |\mathbf{J}_g| \, dg}{\int_{\mathcal{G}} p(g(\mathbf{y}); \mathcal{H}_0) |\mathbf{J}_g| \, dg},$$

where $\mathcal{G}$ is the group of transformations under which the test is invariant, $|\mathbf{J}_g|$ denotes the absolute value of the Jacobian of the transformations $g(\cdot) \in \mathcal{G}$ and dg is an invariant group measure. The group of invariant transformations for our problem is

$$\mathcal{G} = \{g : \mathbf{x}[n] \to g(\mathbf{x}[n]) = \mathbf{P}\mathbf{G}\mathbf{x}[n], \mathbf{P} \in \mathbb{P}, \mathbf{G} \in \mathbb{D}\},$$

where $\mathbb{D}$ is the set of diagonal matrices and $\mathbb{P}$ the set of permutation matrices.[2] In words, we focus on tests preserving the invariances of the testing problem in (2), which consist of permutations and scalings of the elements of $\mathbf{x}[n]$. Moreover, we have to point out that considering just one of these invariances does not allow us to derive the LMPIT.

[1] Step (v) is only necessary if the likelihood ratio depends on the unknown parameters. Otherwise, the density ratio of the maximal invariants would provide the uniformly most powerful invariant test (UMPIT) statistic.

[2] There is an additional invariance, right multiplication of the data matrix by unitary matrix, but it is not taken into account since it is removed in the sufficient statistic, namely $\hat{\mathbf{R}}$.

Now, due to the problem invariances, we may assume without loss of generality that

$$\mathbf{S} = \mathbf{R}_1^{-1} = \begin{bmatrix} 1 & s_{1,2} & \cdots & s_{1,L} \\ s_{2,1} & 1 & \cdots & s_{2,L} \\ \vdots & \cdots & \ddots & \vdots \\ s_{L,1} & s_{L,2} & \cdots & 1 \end{bmatrix},$$

and $\hat{\mathbf{D}} = \mathbf{I}$, which yields

$$\hat{\mathbf{R}} = \hat{\mathbf{C}} = \begin{bmatrix} 1 & \hat{c}_{1,2} & \cdots & \hat{c}_{1,L} \\ \hat{c}_{2,1} & 1 & \cdots & \hat{c}_{2,L} \\ \vdots & \cdots & \ddots & \vdots \\ \hat{c}_{L,1} & \hat{c}_{L,2} & \cdots & 1 \end{bmatrix}.$$

Hence, the ratio of densities is given by

$$\mathscr{L} = \frac{\sum_{\mathbb{P}} \int_{\mathbb{D}} |\mathbf{S}|^M |\mathbf{G}|^M \, \mathrm{e}^{-M \operatorname{tr}\left(\mathbf{P}^T \mathbf{S} \mathbf{P} \mathbf{G} \hat{\mathbf{C}} \mathbf{G}^H\right)} \, d\mathbf{G}}{\sum_{\mathbb{P}} \int_{\mathbb{D}} |\mathbf{D}|^{-M} |\mathbf{G}|^M \, \mathrm{e}^{-M \operatorname{tr}\left(\mathbf{P}^T \mathbf{D}^{-1} \mathbf{P} \mathbf{G} \hat{\mathbf{C}} \mathbf{G}^H\right)} \, d\mathbf{G}},$$

and it is straightforward to show that

$$\mathscr{L} \propto \sum_{\mathbb{P}} \int_{\mathbb{D}} |\mathbf{G}|^M \, \mathrm{e}^{-M \operatorname{tr}\left(\mathbf{G}\mathbf{G}^H\right)} \, \mathrm{e}^{-\alpha} \, d\mathbf{G}, \qquad (4)$$

where $\propto$ stands for equality up to additive and multiplicative constant (not depending on data) terms, and

$$\alpha = M \operatorname{tr}\left[\tilde{\mathbf{S}} \mathbf{G} \left(\hat{\mathbf{C}} - \mathbf{I}\right) \mathbf{G}^H\right].$$

with $\tilde{\mathbf{S}} = \mathbf{P}^T \mathbf{S} \mathbf{P}$. Here, we must notice that (4) is a function of the unknown parameters, which allows us to conclude that, in general, there does not exist a uniformly most powerful invariant test (UMPIT). The only exception is the case with $L = 2$ antennas, where the UMPIT coincides with the GLRT, and reduces to the evaluation of the correlation coefficient, i.e., $|\hat{c}_{1,2}|$. Due to the non-existence of an UMPIT, we focus on the challenging low-SNR scenario, which is particularly important in CR networks. In this region we have $\mathbf{S} \simeq \mathbf{I} \Leftrightarrow \alpha \simeq 0$ and, using a second order Taylor's series approximation of $\mathrm{e}^{-\alpha}$ around $\alpha = 0$, eq. (4) may be approximated as follows

$$\mathscr{L} \propto \sum_{\mathbb{P}} \int_{\mathbb{D}} |\mathbf{G}|^M \, \mathrm{e}^{-M \operatorname{tr}\left(\mathbf{G}\mathbf{G}^H\right)} \left(\alpha^2 - 2\alpha\right) d\mathbf{G}. \qquad (5)$$

Due to the lack of space, the technical details of the derivation are ommited and will be presented in a forthcoming journal version of the paper. Nevertheless, the proof is sketched for completeness. First, noting the symmetries of the terms in α, it may be shown that the linear term in (5) is zero. By similar arguments, it is easily shown that

$$\mathscr{L} \propto \sum_{\mathbb{P}} \int_{\mathbb{D}} |\mathbf{G}|^M \, \mathrm{e}^{-M \operatorname{tr}\left(\mathbf{G}\mathbf{G}^H\right)} \alpha^2 d\mathbf{G} = \Delta \sum_{\substack{k,m=1 \\ k \neq m}}^{L} |\hat{c}_{m,k}|^2,$$

where

$$\Delta = \sum_{\mathbb{P}} |\tilde{s}_{k,m}|^2 \int_{\mathbb{D}} |\mathbf{G}|^M \, \mathrm{e}^{-M \operatorname{tr}\left(\mathbf{G}\mathbf{G}^H\right)} |g_m|^2 |g_k|^2 d\mathbf{G},$$

which does not depend on the indexes $\{k, m\}$ and, therefore, it has been possible to take it out of the summation. Finally, taking into account that $\sum_{k=1}^{L} |\hat{c}_{k,k}|^2 = L$, we can conclude that the LMPIT statistic is given by

$$\mathscr{L} \propto \sum_{k,m=1}^{L} |\hat{c}_{m,k}|^2 = \|\hat{\mathbf{C}}\|_{\mathrm{F}}^2 \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \eta_{\mathrm{L}}. \qquad (6)$$

4.1. Further comments

We must point out that the LMPIT given by (6) has been previously proposed as an ad-hoc detector in [4], where the authors introduced the Frobenius norm of the coherence matrix as a computationally cheaper detector, in comparison to the GLRT. However, we have shown that (6) is actually the optimal detector, among those invariant, in the low SNR regime. Additionally, it was evidenced by simulations in [10] that the detector based on the Frobenius norm works better than the GLRT for low SNR and/or low sample size.

5. NUMERICAL RESULTS

The performance of the LMPIT is evaluated in this section by means of Monte Carlo simulations. The covariance matrices, $\mathbf{D}$ and $\mathbf{R}_s$, are fixed during the experiment and the SNR is defined as follows

$$\text{SNR (dB)} = 10 \log_{10} \frac{\operatorname{tr}(\mathbf{R}_s)}{\operatorname{tr}(\mathbf{D})}.$$

In the first experiment, we obtain the receiver operating characteristic (ROC) curves for different number of samples and a fixed SNR. We have considered the following parameters: $L = 6$ antennas, $\text{SNR} = 0\,\text{dB}$, the signal covariance matrix is $\mathbf{R}_s = \mathbf{F} \operatorname{diag}(5, 1.4, 1.3, 1.2, 1.1, 1) \mathbf{F}^H$, where $\mathbf{F}$ is the 6×6 Fourier matrix, the noise levels at each antenna are $0.5\,\text{dB}$, $-1\,\text{dB}$, $0\,\text{dB}$, $1\,\text{dB}$, $-0.5\,\text{dB}$ and $0\,\text{dB}$; and $M = 10, 30, 50, 70,$ and 90. The ROC curve[3] for each value of M is shown in Figure 1, where we can see that the LMPIT outperforms the GLRT for all values of M in this scenario (close hypotheses or low SNR). These differences will be even larger for greater values of L. However, we have not considered those scenarios since they are not typical in CR networks.

The second experiment analyzes the performance of the GLRT and the LMPIT for different values of the SNR. Figure 2 shows the probability of missed detection (p_m) for a fixed probability of false alarm $p_{fa} = 0.05$ and $M = 30$. The

[3] Since the GLRT and the LMPIT are invariant tests, the thresholds may be obtained for $\mathbf{D} = \mathbf{I}$ by means of simulations, and those thresholds will be valid for any choice of $\mathbf{D}$.

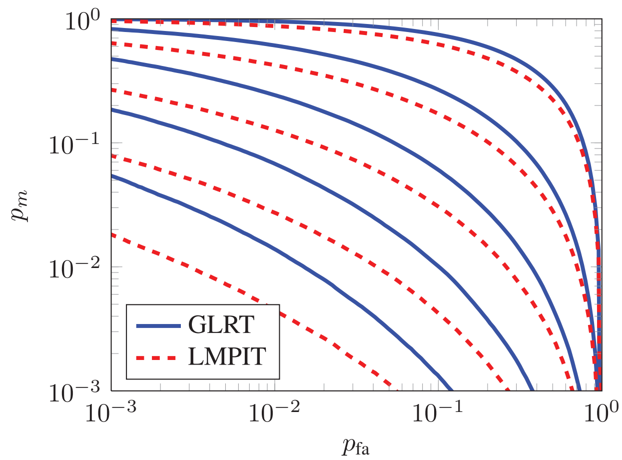

Fig. 1. ROC curves for different values of M in an experiment with $L = 6$ antennas and SNR $= 0\,\mathrm{dB}$.

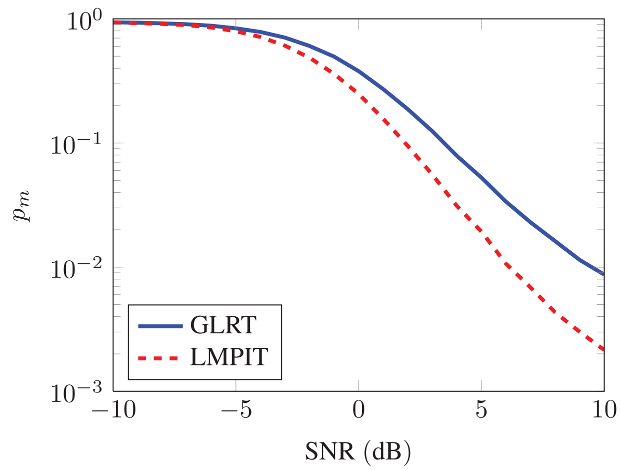

Fig. 2. Probability of missed detection for $p_{fa} = 0.05$ and $M = 30$.

remaining parameters are the same as in the previous experiment. As expected, the performance of the LMPIT is better for these small/moderate values of the SNR.

6. CONCLUSIONS

We have derived the locally most powerful invariant test (LMPIT) for testing whether a set of measurements are spatially correlated. That is, focusing on the family of tests preserving the invariances of the detection problem, we have obtained the optimal test in the case of close hypotheses. Interestingly, it turns out that the LMPIT coincides with a previously proposed heuristic detector. In the derivation of the test, Wijsman's theorem allows us to obtain the density ratio without deriving the distributions of the maximal invariant statistic. Finally, some simulation results illustrate the better performance of the LMPIT for small and moderate values of the SNR.

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