2010 IEEE International Workshop on Machine Learning for Signal Processing (MLSP 2010)
August 29 – September 1, 2010, Kittilä, Finland

IMPROPERNESS MEASURES FOR QUATERNION RANDOM VECTORS

Javier Vía, David Ramírez, Ignacio Santamaría and Luis Vielva

Dept. of Communications Engineering. University of Cantabria. Spain
e-mail: {jvia,ramirezgd,nacho,luis}@gtas.dicom.unican.es

ABSTRACT

It has been recently proved that the two main kinds of quaternion improperness require two different kinds of widely linear processing. In this work, we show that these definitions satisfy some important properties, which include the invariance to quaternion linear transformations and right Clifford translations, as well as some clear connections with the case of proper complex vectors. Moreover, we introduce a new kind of quaternion properness, which clearly relates the two previous definitions, and propose three measures for the degree of improperness of a quaternion vector. The proposed measures are based on the Kullback-Leibler divergence between two zero-mean quaternion Gaussian distributions, one of them with the actual augmented covariance matrix, and the other with its closest proper version. These measures allow us to quantify the entropy loss due to the improperness of the quaternion vector, and they admit an intuitive geometrical interpretation based on Kullback-Leibler projections onto sets of proper augmented covariance matrices.

1. INTRODUCTION

In the last years, quaternion [1, 2] signal processing has received increasing interest due to its successful application to image processing [3], wind modeling [4], and design of space-time block codes [5, 6], which has motivated the extension of several signal processing techniques to the case of quaternionic signals [7, 8]. However, unlike the complex case, only a few works have considered the fundamental problem of analyzing the properness of quaternion random vectors [9–11] and its implication on the structure of the optimal linear processing [12].

In this work we review the two main properness definitions, and show that they can be related by means of a third kind of quaternion properness. Unlike previous approaches, which were based on the invariance of the quaternion second-order statistics (SOS) to left Clifford translations [13], the proposed definitions are directly based on the cancelation of three complementary covariance matrices, and they naturally result in invariances to quaternion linear transformations and right Clifford translations.

In order to quantify the degree of improperness of a quaternion random vector, we propose three improperness measures based on the Kullback-Leibler (KL) divergence [14] between zero-mean quaternion Gaussian distributions. Specifically, the properness measures are given by the KL divergence between the actual *augmented covariance matrix* and its closest (in the KL sense) proper version (for the required kind of properness). Thus, the proposed measures provide the entropy loss due to the improperness of the quaternion vector, or equivalently, the *mutual information* among the quaternion vector and its involutions over three pure unit quaternions. Moreover, the improperness measures admit an intuitive geometrical interpretation based on KL projections onto sets of proper augmented covariance matrices, which also corroborates the relationship among the three kinds of quaternion properness.

Finally, although it is beyond the scope of this paper, the proposed properness measures should find direct application to the problem of testing for the properness of a quaternion random vector or signal. These statistical tests will allow us to determine the most appropriate kind of linear processing, namely, conventional linear processing, *semi-widely* linear processing, or *full-widely* linear processing [12, 15].

This work was supported by the Spanish Government, Ministerio de Ciencia e Innovación (MICINN), under projects MultiMIMO (TEC2007-68020-C04-02) and COMONSENS (CSD2008-00010, CONSOLIDER-INGENIO 2010 Program). Additionally, the work of the two first authors was supported by the Spanish Government, Ministerio de Educación, under grants JC2009-00140 and AP2006-2965.

2. PRELIMINARIES

Throughout this paper we will use bold-faced upper case letters to denote matrices, bold-faced lower case letters for column vectors, and light-faced lower case letters for scalar quantities. Superscripts $(\cdot)^T$ and $(\cdot)^H$ denote transpose and Hermitian (i.e., transpose and quaternion conjugate), respectively. The notation $\mathbf{A} \in \mathbb{F}^{n\times m}$ denotes that $\mathbf{A}$ is a $n \times m$ matrix with entries in $\mathbb{F}$, where $\mathbb{F}$ can be $\mathbb{R}$, the field of real numbers, $\mathbb{C}$, the field of complex numbers, or $\mathbb{H}$, the skew-field of quaternion numbers. $\text{Tr}(\mathbf{A})$ denotes the trace of matrix $\mathbf{A}$, $\otimes$ is the Kronecker product, $\mathbf{I}_n$ is the identity matrix of dimension n, and $\mathbf{0}_{m\times n}$ denotes the $m \times n$ zero matrix. $\mathbf{A}^{1/2}$ (respectively $\mathbf{A}^{-1/2}$) is the Hermitian square root of the Hermitian matrix $\mathbf{A}$ (respectively $\mathbf{A}^{-1}$). Finally, E is the expectation operator, and in general $\mathbf{R}_{\mathbf{a},\mathbf{b}}$ is the cross-correlation matrix for vectors $\mathbf{a}$ and $\mathbf{b}$, i.e., $\mathbf{R}_{\mathbf{a},\mathbf{b}} = E\mathbf{a}\mathbf{b}^H$.

2.1. Properness of Complex Random Vectors

Let us consider a n-dimensional zero-mean[1] complex vector $\mathbf{x} = \mathbf{r}_1 + i\mathbf{r}_i$ with real and imaginary parts $\mathbf{r}_1 \in \mathbb{R}^{n\times 1}$ and $\mathbf{r}_i \in \mathbb{R}^{n\times 1}$ respectively. The second-order statistics (SOS) of $\mathbf{x}$ are given by the covariance $\mathbf{R}_{\mathbf{x},\mathbf{x}} = E\mathbf{x}\mathbf{x}^H$ and complementary covariance $\mathbf{R}_{\mathbf{x},\mathbf{x}^*} = E\mathbf{x}\mathbf{x}^T$ matrices [16, 17], or equivalently by the $2n \times 2n$ augmented covariance matrix

$$\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = E\bar{\mathbf{x}}\bar{\mathbf{x}}^H = \begin{bmatrix} \mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^*} \\ \mathbf{R}^*_{\mathbf{x},\mathbf{x}^*} & \mathbf{R}^*_{\mathbf{x},\mathbf{x}} \end{bmatrix},$$

[1] In this paper we consider zero-mean vectors for notational simplicity. The extension of the results to non-zero mean vectors is straightforward.

where $\bar{\mathbf{x}} = \left[\mathbf{x}^T, \mathbf{x}^H\right]^T \in \mathbb{C}^{2n\times 1}$ is defined as the augmented complex vector. Thus, $\mathbf{x}$ is said to be *proper* if and only if (iff)

$$\mathbf{R}_{\mathbf{x},\mathbf{x}^*} = \mathbf{0}_{n\times n}, \tag{1}$$

i.e., iff $\mathbf{x}$ is uncorrelated with its complex conjugate $\mathbf{x}^*$. In terms of the real vectors $\mathbf{r}_1$ and $\mathbf{r}_i$, the condition in (1) can be written as

$$\mathbf{R}_{\mathbf{r}_1,\mathbf{r}_1} = \mathbf{R}_{\mathbf{r}_i,\mathbf{r}_i}, \tag{2}$$

$$\mathbf{R}_{\mathbf{r}_1,\mathbf{r}_i} = -\mathbf{R}^T_{\mathbf{r}_1,\mathbf{r}_i}. \tag{3}$$

The properness definition can be easily extended to the case of two complex random vectors $\mathbf{x} \in \mathbb{C}^{n\times 1}$ and $\mathbf{y} \in \mathbb{C}^{m\times 1}$. In particular, $\mathbf{x}$ and $\mathbf{y}$ are *jointly-proper* iff they are proper and cross-proper, or equivalently, iff the composite vector $\left[\mathbf{x}^T, \mathbf{y}^T\right]^T$ is proper [17, 18].

From a practical point of view, the (joint)-properness of random vectors translates into the optimality of *conventional* linear processing, that is, operations of the form $\mathbf{y} = \mathbf{F}_1^H\mathbf{x}$, with $\mathbf{F}_1 \in \mathbb{C}^{n\times m}$. However, in the general case of improper complex vectors, the optimal linear processing (which is referred to as *widely-linear* processing) takes the form

$$\mathbf{y} = \mathbf{F}_{\bar{\mathbf{x}}}^H\bar{\mathbf{x}} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_i^H\mathbf{x}^*,$$

with $\mathbf{F}_{\bar{\mathbf{x}}} = \begin{bmatrix}\mathbf{F}_1^T & \mathbf{F}_i^T\end{bmatrix}^T \in \mathbb{C}^{2n\times m}$.

2.2. Quaternion Algebra

Quaternions are four-dimensional hypercomplex numbers defined as

$$x = r_1 + \eta r_\eta + \eta' r_{\eta'} + \eta'' r_{\eta''},$$

where $r_1, r_\eta, r_{\eta'}, r_{\eta''} \in \mathbb{R}$ are four real numbers, and the three imaginary units[2] (η, η', η'') satisfy

$$\begin{aligned}
&\eta\eta' = \eta'' = -\eta'\eta,\\
&\eta'\eta'' = \eta = -\eta''\eta',\\
&\eta''\eta = \eta' = -\eta\eta'',\\
&\eta^2 = \eta'^2 = \eta''^2 = \eta\eta'\eta'' = -1.
\end{aligned}$$

Quaternions form a noncommutative normed division algebra $\mathbb{H}$, i.e., for $x, y \in \mathbb{H}$, $xy \neq yx$ in general. The conjugate of a quaternion x is $x^* = r_1 - \eta r_\eta - \eta' r_{\eta'} - \eta'' r_{\eta''}$, and the inner product of two quaternions x, y is defined as the real part of xy^*. Two quaternions are orthogonal iff their inner product is zero, and the norm of a quaternion x is $|x| = \sqrt{xx^*} = \sqrt{r_1^2 + r_\eta^2 + r_{\eta'}^2 + r_{\eta''}^2}$. Furthermore, we say that ν is a pure unit quaternion iff $\nu^2 = -1$ (i.e., iff $|\nu| = 1$ and its real part is zero). Quaternions also admit the Euler representation

$$x = |x|e^{\nu\theta} = |x|\left(\cos\theta + \nu\sin\theta\right),$$

where ν is a pure unit quaternion and $\theta \in \mathbb{R}$ is the angle (or argument) of the quaternion. Thus, given an angle θ and a pure unit quaternion ν, we can define the left (respectively right) Clifford translation of $x \in \mathbb{H}$ as the product $e^{\nu\theta}x$ (resp. $xe^{\nu\theta}$) [13].

[2]A particular choice of the imaginary axes is the canonical basis $\{i, j, k\}$. However, in this paper we use the more general representation $\{\eta, \eta', \eta''\}$.

The involution of a quaternion x over a pure unit quaternion ν is

$$x^{(\nu)} = -\nu x\nu,$$

and it represents a rotation of angle π in the imaginary plane orthogonal to $\{1, \nu\}$. Alternatively, quaternions can be represented by means of the *Cayley-Dickson* construction

$$x = a_1 + \eta'' a_2, \qquad x = b_1 + \eta b_2, \qquad x = c_1 + \eta' c_2, \tag{4}$$

where

$$\begin{aligned}
a_1 &= r_1 + \eta r_\eta, & b_1 &= r_1 + \eta' r_{\eta'}, & c_1 &= r_1 + \eta'' r_{\eta''},\\
a_2 &= r_{\eta''} + \eta r_{\eta'}, & b_2 &= r_\eta + \eta' r_{\eta''}, & c_2 &= r_{\eta'} + \eta'' r_\eta,
\end{aligned}$$

can be seen as complex numbers in the planes spanned by $\{1, \eta\}$, $\{1, \eta'\}$ and $\{1, \eta''\}$, respectively.

3. PROPERNESS OF QUATERNION VECTORS

In this section we review the definitions of proper quaternion vectors presented in [12], pointing out their differences with previous works. Additionally, we present some important properties of the proposed definitions, and introduce a third kind of quaternion properness, which allows us to establish a clear relationship between the two main types of quaternion properness.

3.1. Properness Definitions

Given a n-dimensional quaternion random vector $\mathbf{x} = \mathbf{r}_1 + \eta\mathbf{r}_\eta + \eta'\mathbf{r}_{\eta'} + \eta''\mathbf{r}_{\eta''}$, the properness definitions in [12] are based on the augmented covariance matrix, which contains all the second-order statistics (SOS) of the quaternion vector. Specifically, the augmented covariance matrix is defined as

$$\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \begin{bmatrix}
\mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}\\
\mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}^{(\eta')}}\\
\mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}} & \mathbf{R}^{(\eta')}_{\mathbf{x},\mathbf{x}^{(\eta)}}\\
\mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta'')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta')}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}^{(\eta)}} & \mathbf{R}^{(\eta'')}_{\mathbf{x},\mathbf{x}}
\end{bmatrix}, \tag{5}$$

where $\bar{\mathbf{x}} = \left[\mathbf{x}^T, \mathbf{x}^{(\eta)T}, \mathbf{x}^{(\eta')T}, \mathbf{x}^{(\eta'')T}\right]^T$ is the augmented quaternion vector, which can be easily obtained from the real vector

$$\mathbf{r}_\mathbf{x} = \left[\mathbf{r}_1^T, \mathbf{r}_\eta^T, \mathbf{r}_{\eta'}^T, \mathbf{r}_{\eta''}^T\right]^T$$

as $\bar{\mathbf{x}} = 2\mathbf{T}_n\mathbf{r}_\mathbf{x}$, where

$$\mathbf{T}_n = \frac{1}{2}\begin{bmatrix}
+1 & +\eta & +\eta' & +\eta''\\
+1 & +\eta & -\eta' & -\eta''\\
+1 & -\eta & +\eta' & -\eta''\\
+1 & -\eta & -\eta' & +\eta''
\end{bmatrix} \otimes \mathbf{I}_n,$$

is a unitary quaternion operator, i.e., $\mathbf{T}_n^H\mathbf{T}_n = \mathbf{I}_{4n}$.

From (5) we can readily identify the covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}} = E\mathbf{x}\mathbf{x}^H$ and three complementary covariance matrices

$$\begin{aligned}
\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}} &= E\mathbf{x}\mathbf{x}^{(\eta)H},\\
\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}} &= E\mathbf{x}\mathbf{x}^{(\eta')H},\\
\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}} &= E\mathbf{x}\mathbf{x}^{(\eta'')H}.
\end{aligned}$$

Thus, we can introduce the two main kinds of quaternion properness, which are closely related to (but different from) the previous definitions in [9–11].

Definition 1 ($\mathbb{Q}$-Properness [12]) *A quaternion random vector* $\mathbf{x}$ *is* $\mathbb{Q}$*-proper iff the three complementary covariance matrices* $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$, $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ *and* $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ *vanish.*

Definition 2 ($\mathbb{C}^{\eta}$-Properness [12]) *A quaternion random vector* $\mathbf{x}$ *is* $\mathbb{C}^{\eta}$*-proper iff the complementary covariance matrices* $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta')}}$ *and* $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta'')}}$ *vanish.*

Finally, we introduce a third (and completely new) kind of quaternion properness.

Definition 3 ($\mathbb{R}^{\eta}$-Properness) *A quaternion random vector* $\mathbf{x}$ *is* $\mathbb{R}^{\eta}$*-proper iff the complementary covariance matrix* $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\eta)}}$ *vanishes.*

Obviously, this intuitive kind of quaternion properness can be seen as the *difference* between $\mathbb{C}^{\eta}$ and $\mathbb{Q}$ properness. Roughly speaking, we can say that $\mathbb{R}^{\eta}$-properness is all what a $\mathbb{C}^{\eta}$-proper vector needs to become $\mathbb{Q}$-proper.

3.2. Main Properties

The main properties of these properness definitions can be summarized as follows.

3.2.1. Independence of the Orthogonal Basis $\{\eta, \eta', \eta''\}$

It is clear that the $\mathbb{R}^{\eta}$-properness definition depends on the pure unit quaternion η, but it is independent of η' and η''. The same holds for the $\mathbb{C}^{\eta}$-properness definition, i.e., a quaternion random vector $\mathbf{x}$ is $\mathbb{C}^{\eta}$-proper iff the complementary covariance matrix $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\nu)}}$ vanishes for all pure unit quaternions ν orthogonal to η [12]. Finally, it is easy to prove that the $\mathbb{Q}$-properness definition is completely independent of the basis $\{\eta, \eta', \eta''\}$, i.e., $\mathbf{x}$ is $\mathbb{Q}$-proper iff $\mathbf{R}_{\mathbf{x},\mathbf{x}^{(\nu)}} = \mathbf{0}_{n\times n}$ for all pure unit quaternions ν [12].

3.2.2. Invariance to Linear Transformations

It can be easily checked that the structure (location of zero complementary covariance matrices) of the augmented covariance matrix $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ is invariant to linear transformations of the form $\mathbf{F}_1^H\mathbf{x}$, with $\mathbf{F}_1 \in \mathbb{H}^{n\times m}$. Thus, since the three proposed properness definitions are based on the cancelation of the complementary covariance matrices, they are invariant to quaternion linear transformations, i.e., if $\mathbf{x}$ is $\mathbb{Q}$-proper (respectively $\mathbb{C}^{\eta}$ or $\mathbb{R}^{\eta}$ proper), then $\mathbf{F}_1^H\mathbf{x}$ is also $\mathbb{Q}$-proper (resp. $\mathbb{C}^{\eta}$ or $\mathbb{R}^{\eta}$ proper). This invariance is very desirable in signal processing applications, and justifies[3] the use of the proposed properness definitions instead of previous approaches based on invariances to left Clifford translations [9–11]. Finally, it is also easy to prove that the proposed $\mathbb{C}^{\eta}$-properness definition is invariant to the more general *semi-widely linear* transformations $\mathbf{F}_1^H\mathbf{x} + \mathbf{F}_{\eta}^H\mathbf{x}^{(\eta)}$, with $\mathbf{F}_1, \mathbf{F}_{\eta} \in \mathbb{H}^{n\times m}$ [12].

3.2.3. Relationship with the Complex Case

Using the Cayley-Dickson representations in (4), it can be proved that a quaternion vector $\mathbf{x} = \mathbf{a}_1 + \eta''\mathbf{a}_2$ is $\mathbb{C}^{\eta}$-proper iff

$$\mathbf{R}_{\mathbf{a}_1,\mathbf{a}_1^*} = \mathbf{R}_{\mathbf{a}_2,\mathbf{a}_2^*} = \mathbf{R}_{\mathbf{a}_1,\mathbf{a}_2^*} = \mathbf{0}_{n\times n}.$$

In other words, $\mathbf{x}$ is $\mathbb{C}^{\eta}$-proper iff it can be represented by means of two jointly-proper complex vectors ($\mathbf{a}_1 = \mathbf{r}_1 + \eta\mathbf{r}_{\eta}$ and $\mathbf{a}_2 = \mathbf{r}_{\eta''} + \eta\mathbf{r}_{\eta'}$) in the plane spanned by $\{1, \eta\}$. Obviously, a similar (but stronger) result holds for $\mathbb{Q}$-proper vectors. Finally, in the case of $\mathbb{R}^{\eta}$-proper vectors we have that $\mathbf{x}$ is $\mathbb{R}^{\eta}$-proper iff $\mathbf{R}_{\mathbf{a}_1,\mathbf{a}_1} = \mathbf{R}_{\mathbf{a}_2,\mathbf{a}_2}^T$ and $\mathbf{R}_{\mathbf{a}_1,\mathbf{a}_2} = -\mathbf{R}_{\mathbf{a}_1,\mathbf{a}_2}^T$, which resembles the complex properness conditions in eqs. (2) and (3).

3.2.4. Implications on the Optimal Linear Processing

In general, the optimal linear processing of a quaternion vector is *full-widely linear*, which operates on the quaternion vector and its three involutions

$$\mathbf{u} = \mathbf{F}_{\bar{\mathbf{x}}}^H\bar{\mathbf{x}} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_{\eta}^H\mathbf{x}^{(\eta)} + \mathbf{F}_{\eta'}^H\mathbf{x}^{(\eta')} + \mathbf{F}_{\eta''}^H\mathbf{x}^{(\eta'')},$$

with $\mathbf{F}_{\bar{\mathbf{x}}} = \left[\mathbf{F}_1^T, \mathbf{F}_{\eta}^T, \mathbf{F}_{\eta'}^T, \mathbf{F}_{\eta''}^T\right]^T \in \mathbb{H}^{4n\times m}$. However, in [12] it has been proved that, in the case of jointly $\mathbb{Q}$-proper vectors, the principal multivariate statistical analysis techniques reduce to *conventional* linear processing $\mathbf{u} = \mathbf{F}_1^H\mathbf{x}$, whereas in the case of $\mathbb{C}^{\eta}$-proper vectors the optimal processing is *semi-widely linear*

$$\mathbf{u} = \mathbf{F}_{\tilde{\mathbf{x}}}^H\tilde{\mathbf{x}} = \mathbf{F}_1^H\mathbf{x} + \mathbf{F}_{\eta}^H\mathbf{x}^{(\eta)},$$

where $\mathbf{F}_{\tilde{\mathbf{x}}} = [\mathbf{F}_1^T, \mathbf{F}_{\eta}^T]^T \in \mathbb{H}^{2n\times m}$, and $\tilde{\mathbf{x}} = [\mathbf{x}^T, \mathbf{x}^{(\eta)^T}]^T$ is defined as the *semi-augmented quaternion vector*. Finally, although the $\mathbb{R}^{\eta}$-properness definition does not result in a simplification of the optimal linear processing, it establishes a link between the two main kinds of quaternion properness.

3.2.5. SOS Invariance to Right Clifford Translations

Unlike previous approaches, which were based on invariances to left Clifford translations [9–11], the proposed properness definitions naturally result in invariances to right Clifford translations. Here we summarize the three main invariance properties, whose proof can be found in [15].

Property 1 *A quaternion random vector* $\mathbf{x} = \mathbf{a}_1 + \eta''\mathbf{a}_2$ *is* $\mathbb{R}^{\eta}$*-proper iff the covariance* $\mathbf{R}_{\mathbf{a}_1,\mathbf{a}_1}$, $\mathbf{R}_{\mathbf{a}_2,\mathbf{a}_2}$ *and cross-covariance* $\mathbf{R}_{\mathbf{a}_1,\mathbf{a}_2}$ *matrices are invariant to right multiplications by the pure unit quaternion* η''.

Property 2 *A quaternion random vector* $\mathbf{x}$ *is* $\mathbb{C}^{\eta}$*-proper iff its SOS are invariant under right Clifford translations* $\mathbf{x}e^{\eta\theta}$, $\forall\theta \in \mathbb{R}$.

Here we must note that, although the invariance associated to the $\mathbb{R}^{\eta}$-properness definition seems to be much weaker than that of $\mathbb{C}^{\eta}$-proper vectors, this weak invariance is sufficient (and necessary) to guarantee the $\mathbb{Q}$-properness of a $\mathbb{C}^{\eta}$-proper vector. Finally, the last property follows from the fact that $\mathbb{Q}$-properness implies $\mathbb{C}^{\eta}$-properness for all η.

Property 3 *A quaternion random vector* $\mathbf{x}$ *is* $\mathbb{Q}$*-proper iff its SOS are invariant to right Clifford translations* $\mathbf{x}e^{\eta\theta}$ *for all pure unit quaternions* η *and* $\forall\theta \in \mathbb{R}$.

4. IMPROPERNESS MEASURES

In the case of complex random vectors, improperness measures have been proposed in [19]. Here, we extend this idea to quaternion vectors. In particular, given a random vector $\mathbf{x} \in \mathbb{H}^{n\times 1}$ with

[3] It is straightforward to prove that the previous properness definitions are not invariant to quaternion linear transformations of the form $\mathbf{F}_1^H\mathbf{x}$.

Table 1. Probability density function, Entropy, and Kullback-Leibler Divergence of Quaternion Gaussian Vectors.

	Probability density function (pdf)	Entropy	Kullback-Leibler Divergence
Expression from the real vector $\mathbf{r}_\mathbf{x}$	$p_\mathbf{x}(\mathbf{r}_\mathbf{x}) = \frac{\exp\left(-\frac{1}{2}\mathbf{r}_\mathbf{x}^T\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}^{-1}\mathbf{r}_\mathbf{x}\right)}{(2\pi)^{2n}\lvert\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}\rvert^{1/2}}$	$H_\mathbf{x}(\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}) = 2n\ln(2\pi e) + \frac{1}{2}\ln\lvert\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}\rvert$	$D(\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}\Vert\hat{\mathbf{R}}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}) = \frac{1}{2}\ln\left(\frac{\lvert\hat{\mathbf{R}}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}\rvert}{\lvert\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}\rvert}\right) + \frac{1}{2}\text{Tr}\left(\hat{\mathbf{R}}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}^{-1}\mathbf{R}_{\mathbf{r}_\mathbf{x},\mathbf{r}_\mathbf{x}}\right) - 2n$
Expression from the augmented vector $\bar{\mathbf{x}}$	$p_\mathbf{x}(\bar{\mathbf{x}}) = \frac{\exp\left(-\frac{1}{2}\bar{\mathbf{x}}^H\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1}\bar{\mathbf{x}}\right)}{(\pi/2)^{2n}\lvert\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\rvert^{1/2}}$	$H_\mathbf{x}(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}) = 2n\ln\left(\frac{\pi e}{2}\right) + \frac{1}{2}\ln\lvert\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\rvert$	$D(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\Vert\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}) = \frac{1}{2}\ln\left(\frac{\lvert\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\rvert}{\lvert\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\rvert}\right) + \frac{1}{2}\text{Tr}\left(\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1/2}\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}^{-1/2}\right) - 2n$
Expression from the semi-augmented vector $\tilde{\mathbf{x}}$ ($\mathbb{C}^\eta$-proper case)	$p_\mathbf{x}(\tilde{\mathbf{x}}) = \frac{\exp\left(-\tilde{\mathbf{x}}^H\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}^{-1}\tilde{\mathbf{x}}\right)}{(\pi/2)^{2n}\lvert\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}\rvert}$	$H_\mathbf{x}(\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}) = 2n\ln\left(\frac{\pi e}{2}\right) + \ln\lvert\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}\rvert$	$D(\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}\Vert\hat{\mathbf{R}}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}) = \ln\left(\frac{\lvert\hat{\mathbf{R}}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}\rvert}{\lvert\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}\rvert}\right) + \text{Tr}\left(\hat{\mathbf{R}}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}^{-1/2}\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}\hat{\mathbf{R}}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}^{-1/2}\right) - 2n$
Expression from the vector $\mathbf{x}$ ($\mathbb{Q}$-proper case)	$p_\mathbf{x}(\mathbf{x}) = \frac{\exp\left(-2\mathbf{x}^H\mathbf{R}_{\mathbf{x},\mathbf{x}}^{-1}\mathbf{x}\right)}{(\pi/2)^{2n}\lvert\mathbf{R}_{\mathbf{x},\mathbf{x}}\rvert^2}$	$H_\mathbf{x}(\mathbf{R}_{\mathbf{x},\mathbf{x}}) = 2n\ln\left(\frac{\pi e}{2}\right) + 2\ln\lvert\mathbf{R}_{\mathbf{x},\mathbf{x}}\rvert$	$D(\mathbf{R}_{\mathbf{x},\mathbf{x}}\Vert\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}}) = 2\ln\left(\frac{\lvert\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}}\rvert}{\lvert\mathbf{R}_{\mathbf{x},\mathbf{x}}\rvert}\right) + 2\text{Tr}\left(\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}}^{-1/2}\mathbf{R}_{\mathbf{x},\mathbf{x}}\hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}}^{-1/2}\right) - 2n$

augmented covariance matrix $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$, we propose to use the following improperness measure

$$\mathcal{P} = \min_{\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\in\mathcal{R}} D\left(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\Vert\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\right),$$

where $\mathcal{R}$ denotes the set of proper augmented covariance matrices (with the assumed properness), and $D(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\Vert\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}})$ is the Kullback-Leibler divergence between two zero-mean quaternion Gaussian distributions with augmented covariance matrices $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ and $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$.

The probability density function (pdf) of quaternion Gaussian vectors can be easily obtained from the pdf of the real vector $\mathbf{r}_\mathbf{x}$, and it can be simplified in the case of $\mathbb{C}^\eta$-proper or $\mathbb{Q}$-proper vectors. Table 1 shows the pdf, entropy, and Kullback-Leibler divergence expressions for quaternion Gaussian vectors.

4.1. Measure of $\mathbb{Q}$-Improperness

Let us start our analysis by the strongest kind of quaternion properness. The set $\mathcal{R}_\mathbb{Q}$ of $\mathbb{Q}$-proper augmented covariance matrices is

$$\mathcal{R}_\mathbb{Q} = \left\{\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \middle| \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}} = \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta')}} = \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta'')}} = \mathbf{0}_{n\times n}\right\},$$

and the matrix $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \in \mathcal{R}_\mathbb{Q}$ minimizing $D(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\Vert\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}})$ is

$$\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \mathbf{D}_\mathbb{Q} = \begin{bmatrix} \mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{R}_{\mathbf{x},\mathbf{x}}^{(\eta)} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{R}_{\mathbf{x},\mathbf{x}}^{(\eta')} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{R}_{\mathbf{x},\mathbf{x}}^{(\eta'')} \end{bmatrix}.$$

Thus, the $\mathbb{Q}$-improperness measure reduces to

$$\mathcal{P}_\mathbb{Q} = -\frac{1}{2}\ln\lvert\mathbf{\Phi}_\mathbb{Q}\rvert,$$

where we have defined $\mathbf{\Phi}_\mathbb{Q} = \mathbf{D}_\mathbb{Q}^{-\frac{1}{2}}\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}\mathbf{D}_\mathbb{Q}^{-\frac{1}{2}}$ as the $\mathbb{Q}$-coherence matrix. Furthermore, we can easily check that $\mathcal{P}_\mathbb{Q}$ is non-negative, invariant under quaternion linear transformations of the form $\mathbf{F}_1^H\mathbf{x}$, and it can be rewritten as

$$\mathcal{P}_\mathbb{Q} = H_\mathbf{x}(\mathbf{D}_\mathbb{Q}) - H_\mathbf{x}(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}) = H_\mathbf{x}(\mathbf{R}_{\mathbf{x},\mathbf{x}}) - H_\mathbf{x}(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}),$$

which represents the entropy loss due to the improperness of $\mathbf{x}$. That is, $\mathcal{P}_\mathbb{Q}$ can be seen as a measure of the *mutual information* among the random vectors $\mathbf{x}$, $\mathbf{x}^{(\eta)}$, $\mathbf{x}^{(\eta')}$ and $\mathbf{x}^{(\eta'')}$.

4.2. Measure of $\mathbb{C}^\eta$-Improperness

In this case, the set of $\mathbb{C}^\eta$-proper augmented covariance matrices is

$$\mathcal{R}_{\mathbb{C}^\eta} = \left\{ \hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} | \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta')}} = \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta'')}} = \mathbf{0}_{n\times n} \right\},$$

and the matrix $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \in \mathcal{R}_{\mathbb{C}^\eta}$ minimizing $D(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \| \hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}})$ is

$$\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} = \mathbf{D}_{\mathbb{C}^\eta} = \begin{bmatrix} \mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}} & \mathbf{0}_{2n\times 2n} \\ \mathbf{0}_{2n\times 2n} & \mathbf{R}^{(\eta')}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}} \end{bmatrix}.$$

Therefore, the $\mathbb{C}^\eta$-improperness measure reduces to

$$\mathcal{P}_{\mathbb{C}^\eta} = -\frac{1}{2} \ln |\mathbf{\Phi}_{\mathbb{C}^\eta}|,$$

where $\mathbf{\Phi}_{\mathbb{C}^\eta} = \mathbf{D}_{\mathbb{C}^\eta}^{-\frac{1}{2}} \mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \mathbf{D}_{\mathbb{C}^\eta}^{-\frac{1}{2}}$ is defined as the $\mathbb{C}^\eta$-coherence matrix. Furthermore, it is easy to prove that $\mathcal{P}_{\mathbb{C}^\eta}$ is non-negative, invariant to semi-widely linear transformations $\mathbf{F}_1^H \mathbf{x} + \mathbf{F}_\eta^H \mathbf{x}^{(\eta)}$, and it also represents the entropy loss due to the $\mathbb{C}^\eta$-improperness of $\mathbf{x}$, i.e.

$$\mathcal{P}_{\mathbb{C}^\eta} = H_{\mathbf{x}}(\mathbf{D}_{\mathbb{C}^\eta}) - H_{\mathbf{x}}(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}) = H_{\mathbf{x}}(\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}) - H_{\mathbf{x}}(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}).$$

Additionally, we can rewrite the semi-augmented quaternion vector $\tilde{\mathbf{x}} = [\mathbf{x}^T, \mathbf{x}^{(\eta)^T}]^T$ in terms of the Cayley-Dickson representation

$$\underbrace{\begin{bmatrix} \mathbf{x} \\ \mathbf{x}^{(\eta)} \end{bmatrix}}_{\tilde{\mathbf{x}}} = \underbrace{\left(\begin{bmatrix} 1 & \eta'' \\ 1 & -\eta'' \end{bmatrix} \otimes \mathbf{I}_{n,n} \right)}_{\mathbf{L}} \underbrace{\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}}_{\mathbf{a}},$$

which allows us to rewrite the $\mathbb{C}^\eta$-improperness measure as

$$\mathcal{P}_{\mathbb{C}^\eta} = -\frac{1}{2} \ln |\mathbf{\Phi}_{\tilde{\mathbf{a}}}|,$$

where $\mathbf{\Phi}_{\tilde{\mathbf{a}}} = \mathbf{D}_{\tilde{\mathbf{a}}}^{-\frac{1}{2}} \mathbf{R}_{\tilde{\mathbf{a}},\tilde{\mathbf{a}}} \mathbf{D}_{\tilde{\mathbf{a}}}^{-\frac{1}{2}}$ is the coherence matrix for the complex vector $\tilde{\mathbf{a}} = \left[\mathbf{a}^T, \mathbf{a}^H\right]^T$, and

$$\mathbf{D}_{\tilde{\mathbf{a}}} = \begin{bmatrix} \mathbf{R}_{\mathbf{a},\mathbf{a}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{R}^*_{\mathbf{a},\mathbf{a}} \end{bmatrix}.$$

Therefore, the $\mathbb{C}^\eta$-improperness measure is also the improperness measure of the complex vector $\mathbf{a}$ [19], which is a measure of the degree of joint-improperness of the complex vectors $\mathbf{a}_1$, $\mathbf{a}_2$. That is, as previously pointed out, the $\mathbb{C}^\eta$-properness of a quaternion vector $\mathbf{x}$ can be seen as the joint-properness of the complex vectors in the Cayley-Dickson representation $\mathbf{x} = \mathbf{a}_1 + \eta'' \mathbf{a}_2$.

4.3. Measure of $\mathbb{R}^\eta$-Improperness

For the $\mathbb{R}^\eta$-improperness measure, the problem is more involved than in the previous cases. This is due to the fact that, given the set

$$\mathcal{R}_{\mathbb{R}^\eta} = \left\{ \hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} | \hat{\mathbf{R}}_{\mathbf{x},\mathbf{x}^{(\eta)}} = \mathbf{0}_{n\times n} \right\},$$

obtaining the matrix $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \in \mathcal{R}_{\mathbb{R}^\eta}$ that minimizes $D(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \| \hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}})$ is far from trivial, and it is closely related to the problem of maximum likelihood estimation of structured covariance matrices [20, 21].

Here we focus on an alternative and more meaningful measure. In particular, we focus on the $\mathbb{R}^\eta$-improperness of $\mathbb{C}^\eta$-proper vectors. That is, given an augmented covariance matrix $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \in \mathcal{R}_{\mathbb{C}^\eta}$, we look for the closest (in the Kullback-Leibler sense) matrix $\hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \in \mathcal{R}_{\mathbb{R}^\eta}$, and with a slight abuse of notation define

$$\mathcal{P}_{\mathbb{R}^\eta} = D(\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}} \| \hat{\mathbf{R}}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}).$$

Thus, analogously to the previous cases, the $\mathbb{R}^\eta$-improperness measure is

$$\begin{aligned} \mathcal{P}_{\mathbb{R}^\eta} &= -\frac{1}{2} \ln |\mathbf{D}_{\mathbb{Q}}^{-\frac{1}{2}} \mathbf{D}_{\mathbb{C}^\eta} \mathbf{D}_{\mathbb{Q}}^{-\frac{1}{2}}| \\ &= -\ln |\mathbf{D}_{\mathbb{R}^\eta}^{-\frac{1}{2}} \mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}} \mathbf{D}_{\mathbb{R}^\eta}^{-\frac{1}{2}}| = -\ln |\mathbf{\Phi}_{\mathbb{R}^\eta}|, \end{aligned}$$

where

$$\mathbf{D}_{\mathbb{R}^\eta} = \begin{bmatrix} \mathbf{R}_{\mathbf{x},\mathbf{x}} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{R}^{(\eta)}_{\mathbf{x},\mathbf{x}} \end{bmatrix},$$

and $\mathbf{\Phi}_{\mathbb{R}^\eta} = \mathbf{D}_{\mathbb{R}^\eta}^{-\frac{1}{2}} \mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}} \mathbf{D}_{\mathbb{R}^\eta}^{-\frac{1}{2}}$ is the $\mathbb{R}^\eta$-coherence matrix. Finally, the measure $\mathcal{P}_{\mathbb{R}^\eta}$ is non-negative, invariant to linear transformations, and it provides the entropy loss due to the $\mathbb{R}^\eta$-improperness of the ($\mathbb{C}^\eta$-proper) vector $\mathbf{x}$

$$\begin{aligned} \mathcal{P}_{\mathbb{R}^\eta} &= H_{\mathbf{x}}(\mathbf{D}_{\mathbb{Q}}) - H_{\mathbf{x}}(\mathbf{D}_{\mathbb{C}^\eta}) \\ &= H_{\mathbf{x}}(\mathbf{D}_{\mathbb{R}^\eta}) - H_{\mathbf{x}}(\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}) = H_{\mathbf{x}}(\mathbf{R}_{\mathbf{x},\mathbf{x}}) - H_{\mathbf{x}}(\mathbf{R}_{\tilde{\mathbf{x}},\tilde{\mathbf{x}}}), \end{aligned}$$

or equivalently, the mutual information between $\mathbf{x}$ and $\mathbf{x}^{(\eta)}$.

4.4. Relationship among the Improperness Measures

Interestingly, the previous improperness measures satisfy the following relationship

$$\mathcal{P}_{\mathbb{Q}} = \mathcal{P}_{\mathbb{C}^\eta} + \mathcal{P}_{\mathbb{R}^\eta}, \tag{6}$$

which can be seen as a direct consequence of the Pythagorean theorem for exponential families of pdf's [22], and corroborates our intuition about the complementarity of $\mathbb{C}^\eta$ and $\mathbb{R}^\eta$ properness. Moreover, since the $\mathbb{Q}$-improperness measure does not depend on the orthogonal basis $\{1, \eta, \eta', \eta''\}$, $\mathcal{P}_{\mathbb{Q}}$ can be decomposed as (6) for all pure unit quaternions η. In other words, the Kullback-Leibler *distance* from an augmented covariance matrix $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ to the closest $\mathbb{Q}$-proper matrix $\mathbf{D}_{\mathbb{Q}}$ can be calculated as the sum of the *distance* from $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$ to the closest $\mathbb{C}^\eta$-proper matrix $\mathbf{D}_{\mathbb{C}^\eta}$, and the *distance* from $\mathbf{D}_{\mathbb{C}^\eta}$ to the closest $\mathbb{R}^\eta$-proper matrix $\mathbf{D}_{\mathbb{Q}}$. This fact is illustrated in Figure 1 for three orthogonal pure unit quaternions η, η' and η''.

5. CONCLUSIONS

The two main definitions of quaternion-properness have been revisited, showing that they can be related by means of a third kind of properness. We have also shown that, unlike previous approaches, these definitions are invariant to linear quaternion transformations, and they naturally result in the invariance to different kinds of right Clifford translations. More importantly, we have proposed three measures for the degree of improperness of a quaternion random vector. These measures, which are based on the Kullback-Leibler divergence between zero-mean quaternion Gaussian distributions, provide the entropy loss due to the improperness of the quaternion vector, and they also admit an intuitive geometrical interpretation based on Kullback-Leibler projections onto sets of proper augmented covariance matrices. Current research lines include the application of the proposed measures to the problem of testing for the properness of quaternion random vectors.

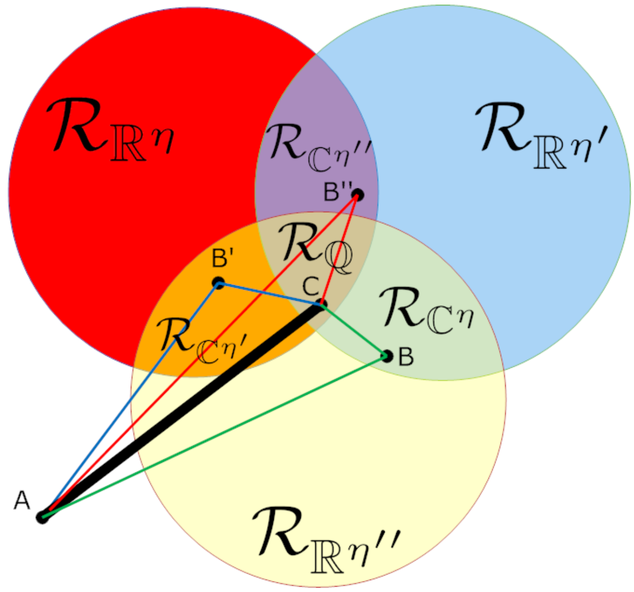

Fig. 1. Illustration of the $\mathbb{Q}$-improperness measure decomposition. The figure shows the sets of $\mathbb{R}$-proper ($\mathcal{R}_{\mathbb{R}^\eta}$, $\mathcal{R}_{\mathbb{R}^{\eta''}}$ and $\mathcal{R}_{\mathbb{R}^{\eta''}}$), $\mathbb{C}$-proper ($\mathcal{R}_{\mathbb{C}^\eta}$, $\mathcal{R}_{\mathbb{C}^{\eta''}}$ and $\mathcal{R}_{\mathbb{C}^{\eta''}}$), and $\mathbb{Q}$-proper ($\mathcal{R}_{\mathbb{Q}}$) augmented covariance matrices. Point A represents a general augmented covariance matrix $\mathbf{R}_{\bar{\mathbf{x}},\bar{\mathbf{x}}}$. B is the closest (in the Kullback-Leibler sense) point to A in $\mathcal{R}_{\mathbb{C}^\eta}$ (matrix $\mathbf{D}_{\mathbb{C}^\eta}$). C (matrix $\mathbf{D}_{\mathbb{Q}}$) is the projection of A onto $\mathcal{R}_{\mathbb{Q}}$, which coincides with the projection of B onto $\mathcal{R}_{\mathbb{R}^\eta}$. The length of the segment $\overline{AC}$ represents the measure $\mathcal{P}_{\mathbb{Q}}$, which is equal to the sum of the lengths of the segments $\overline{AB}$ ($\mathcal{P}_{\mathbb{C}^\eta}$) and $\overline{BC}$ ($\mathcal{P}_{\mathbb{R}^\eta}$). The same interpretation can be made in terms of the points B' and B''.

6. REFERENCES

[1] W. R. Hamilton, "On quaternions," *Proceedings of the Royal Irish Academy*, November 11 1844.

[2] J. P. Ward, *Quaternions and Cayley numbers: Algebra and applications*. Dordrecht, Netherlands: Kluwer Academic, 1997.

[3] T. Bulow and G. Sommer, "Hypercomplex signals–a novel extension of the analytic signal to the multidimensional case," *IEEE Transactions on Signal Processing*, vol. 49, no. 11, pp. 2844–2852, Nov. 2001.

[4] N. ur Rehman and D. Mandic, "Empirical mode decomposition for trivariate signals," *IEEE Transactions on Signal Processing*, vol. 58, no. 3, pp. 1059 –1068, Mar. 2010.

[5] J. Seberry, K. Finlayson, S. Adams, T. Wysocki, T. Xia, and B. Wysocki, "The theory of quaternion orthogonal designs," *IEEE Transactions on Signal Processing*, vol. 56, no. 1, pp. 256–265, Jan. 2008.

[6] S. Sirianunpiboon, A. Calderbank, and S. Howard, "Bayesian analysis of interference cancellation for Alamouti multiplexing," *IEEE Transactions on Information Theory*, vol. 54, no. 10, pp. 4755–4761, Oct. 2008.

[7] S. Miron, N. Le Bihan, and J. Mars, "Quaternion-MUSIC for vector-sensor array processing," *IEEE Transactions on Signal Processing*, vol. 54, no. 4, pp. 1218–1229, Apr. 2006.

[8] C. Took and D. Mandic, "The quaternion LMS algorithm for adaptive filtering of hypercomplex processes," *IEEE Transactions on Signal Processing*, vol. 57, no. 4, pp. 1316–1327, Apr. 2009.

[9] N. N. Vakhania, "Random vectors with values in quaternion hilbert spaces," *Theory of Probability and its Applications*, vol. 43, no. 1, pp. 99–115, 1999.

[10] P. Amblard and N. Le Bihan, "On properness of quaternion valued random variables," in *IMA Conference on Mathematics in Signal Processing*, Cirencester (UK), 2004, pp. 23–26.

[11] N. Le Bihan and S. Buchholz, "Optimal separation of polarized signals by quaternionic neural networks," in *XIV European Signal Processing Conference (EUSIPCO)*, Florence, Italy, 2006.

[12] J. Vía, D. Ramírez, I. Santamaría, and L. Vielva, "Widely and semi-widely linear processing of quaternion vectors," in *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2010)*, Texas, USA, Mar. 2010.

[13] H. S. M. Coxeter, "Quaternions and reflections," *The American Mathematical Monthly*, vol. 53, no. 3, pp. 136–146, 1946.

[14] T. M. Cover and J. A. Thomas, *Elements of information theory*. New York, NY, USA: Wiley-Interscience, 1991.

[15] J. Vía, D. Ramírez, and I. Santamaría, "Properness and widely linear processing of quaternion random vectors," *IEEE Transactions on Information Theory*, vol. 56, no. 7, pp. 3502 –3515, Jul. 2010.

[16] B. Picinbono, "Second-order complex random vectors and normal distributions," *IEEE Transactions on Signal Processing*, vol. 44, no. 10, pp. 2637–2640, Oct 1996.

[17] P. Schreier and L. Scharf, "Second-order analysis of improper complex random vectors and processes," *IEEE Transactions on Signal Processing*, vol. 51, no. 3, pp. 714–725, March 2003.

[18] P. Schreier, "A unifying discussion of correlation analysis for complex random vectors," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1327–1336, April 2008.

[19] ——, "Bounds on the degree of impropriety of complex random vectors," *IEEE Signal Processing Letters*, vol. 15, pp. 190–193, 2008.

[20] J. Burg, D. Luenberger, and D. Wenger, "Estimation of structured covariance matrices," *Proceedings of the IEEE*, vol. 70, no. 9, pp. 963–974, Sep. 1982.

[21] L. Scharf, *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*. New York: Addison-Wesley, 1990.

[22] S.-I. Amari and H. Nagaoka, *Methods of Information Geometry*, A. M. Society, Ed. Oxford University Press, 1993.